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Hecke Algebras, Difference Operators, and Quasi-Symmetric Functions

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We define a new action of the symmetric group and its Hecke algebra on polyno-

polynomial modules of a degenerate quantum group. We use the action of the generic Hecke algebras to define quasi-symmetric and noncommutative analogues of Hall–Littlewood functions. We show that these generalized functions share many combinatorial properties with the classical ones.

Nous introduisons de nouvelles actions du groupe symétrique et de son algèbre de Hecke sur les polynômes, pour lesquelles les invariants sont les polynômes quasi-symétriques. Nous interprétons cette construction en termes de caractères de Demazure d'un groupe quantique dégénéré. Nous utilisons l'action de l'algèbre de Hecke générique pour définir des analogues quasi-symétriques et non commutatifs des fonctions de Hall–Littlewood. Nous montrons que ces fonctions généralisées ont un certain nombre de propriétès communes avec les fonctions classiques.

Contents.

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1. INTRODUCTION

The notion of a symmetric function admits two interesting generalizations: quasi-symmetric functions [12], which are certain partially symmetric polynomials, and noncommutative symmetric functions [11], which are elements of a noncommutative algebra whose abelianization is the usual algebra of symmetric functions.

One of our main results is a Demazure character formula for a degenerate quantum group, which make use of non-standard actions of the symmetric group and its degenerate Hecke algebra. Second, we develop a theory of quasi-symmetric and noncommutative Hall–Littlewood functions. To motivate our construction, let us start with a brief history of the subject.

The Hall–Littlewood symmetric functions were introduced in 1961 by D. E. Littlewood, as a concrete realization of an algebra defined by P. Hall in some unpublished work in the theory of Abelian groups (see the reprint in [14]). Hall's algebra is spanned by isomorphism classes of finite Abelian p-groups, for some fixed prime p. An isomorphism class can be encoded by a partition $\lambda=(\lambda_1, ..., \lambda_r)$ also called the type of the group, recording the exponents in the decomposition

$$G_\lambda \approx \bigoplus_{i=1}^{r} (\mathbb{Z}/p^{\lambda_i}\mathbb{Z}). \tag{1}$$

Let u_λ be the isomorphism class of the above group. The multiplicative structure is defined by

$$u_\alpha u_\beta = \sum_\lambda g^\lambda_{\alpha\beta} u_\lambda, \tag{2}$$

where $g^\lambda_{\alpha\beta}$ is the number of subgroups H of G_λ which are of type α and such that G_λ/H is of type β. Hall showed that these numbers are expressed by polynomials in p, now called Hall polynomials (actually, the Hall algebra had been previously discovered by Steinitz [46], cf. [17]).

A finite Abelian p-group is the same as a finite module over the discrete valuation ring $\mathbb{Z}_p$ (p-adic integers) and the same construction works as well for the rings $\mathbb{K}[[t]]$ where $\mathbb{K}=\mathbb{F}_q$ is a finite field with q elements. Again the $g^\lambda_{\alpha\beta}$ are given by the same polynomials in q. This version of the Hall algebra was used in 1955 by J. A. Green to determine the character table of the finite linear group $GL(n, \mathbb{F}_q)$ [13].

It was known that the Hall algebra was isomorphic to the algebra of symmetric functions, but a basis of symmetric functions having the $g^\lambda_{\alpha\beta}$ as structure constants was not explicitly known, and this was needed to develop a practical algorithm for the calculation of character tables. This is the problem solved by Littlewood: he introduced the symmetric functions

$$P_\mu(x_1, ..., x_n; q) = \prod_{i>0} \frac{1}{[m_i]_q!} \sum_{\sigma \in \mathfrak{S}_n} \sigma \left(x_1^{\mu_1} \cdots x_n^{\mu_n} \prod_{i<j} \frac{x_i - qx_j}{x_i - x_j} \right), \tag{3}$$

where m_i is the multiplicity of i in the partition μ. He proved that

$$u_\lambda = q^{-n(\lambda)} P_\lambda(x; q^{-1}) \tag{4}$$

realize the Hall algebra (here $n(\lambda) = \sum_{i>0} (i-1)\, \lambda_i$). Littlewood further observed that under the specialization $q = -1$, the P_μ reduced to a class of symmetric functions introduced by Schur in 1911 as generating functions of the spin characters of symmetric groups [44].

The Hall–Littlewood functions can be introduced, as in [38], by orthogonalizing the basis of monomial functions with respect to a deformed scalar product. Other deformations with two or more parameters lead to Macdonald's and Kerov's symmetric functions. One denotes by $\{Q_\mu\}$ the dual basis of P_μ for the deformed scalar product. Since P_μ is orthogonal, Q_μ is proportional to P_μ. It is more interesting to look at the dual basis $\{Q'_\mu\}$ of $\{P_\mu\}$ for the ordinary scalar product, for which Schur functions are orthonormal. Q'_μ is a q-analogue of the product h_μ of homogeneous symmetric functions, and the coefficients $K_{\lambda\mu}(q)$ of the expansion

$$Q'_\mu = \sum_\lambda K_{\lambda\mu}(q)\, s_\lambda \tag{5}$$

are q-analogues of the Kotska number $K_{\lambda\mu}$.

H. O. Foulkes conjectured in 1974 [9] that the $K_{\lambda\mu}(q)$ were polynomials with non-negative integer coefficients. This conjecture was proved in 1978 by A. Lascoux and M. P. Schützenberger [29], who introduced a statistic called *charge* on the set $\mathrm{Tab}(\lambda, \mu)$ of Young tableaux of shape λ and weight μ, and proved that

$$K_{\lambda\mu}(q) = \sum_{t \in \mathrm{Tab}(\lambda, \mu)} q^{\mathrm{charge}(t)}. \tag{6}$$

This statistic has now important applications in statistical mechanics, where it appears as the energy of certain quasi-particles in the Bethe Ansatz approach to quantum spin chains [18] or in the corner transfer matrix approach to solvable lattice models [42].

The reciprocal polynomials $\tilde{K}_{\lambda\mu}(q)=q^{n(\mu)}K_{\lambda\mu}(q^{-1})$ are the values of the unipotent characters of $GL(n, \mathbb{F}_q)$ on unipotent classes [45, 36], and their generating function $\tilde{Q}'_\mu=\sum \tilde{K}_{\lambda\mu}(q)\, s_\lambda$ are the graded characteristics of Springer's representations of symmetric groups in the cohomology of unipotent varieties [16]. They appear also in the harmonic polynomial in the flag manifold [23], and there exists a two parameters analogue [10].

Also, the Q'_μ have interesting specializations at roots of unity [24, 25, 38] which appear to be related to the representation theory of quantum affine algebras [28, 33] and of the q-Virasoro algebra [2].

It has been observed in [7] that Littlewood's definition (3) could be interpreted in terms of an action of the Hecke algebra $H_n(q)$ on $\mathbb{C}(q)[x_1, ..., x_n]$ (see also Section 6). This action is obtained by lifting to $H_n(q)$ an action of the degenerate algebra $H_n(0)$, defined by means of the elementary symmetrizing operators involved in Demazure's character formula [5].

This point of view allows us to develop an analogous theory in the context of two recent generalizations of symmetric functions: Quasi-symmetric functions [12] and noncommutative symmetric functions [11]. These objects build up two Hopf algebras which are dual to each other [41] and are related to the representation theory of the degenerate Hecke algebra $H_n(0)$ in the same way as ordinary symmetric functions are related to the symmetric groups [20]. In particular Gessel's fundamental quasi-symmetric functions F_I have been shown in [22] to be the characters of the irreducible polynomial representations of a degenerate quantum group $\mathcal{U}_0(gl_N)$.

The starting point of this paper is the following: There is a non-standard action of the symmetric group on polynomials whose invariant are exactly the quasi-symmetric polynomials (Subsection 3.1). With this action, the isobaric divided differences define an action of the degenerate Hecke algebra (Subsection 3.2). We interpret the action of $\mathfrak{S}_n$ as a the action of the Weyl group and the action of $H_n(0)$ as Demazure operators for the degenerate quantum group $\mathcal{U}_0(gl_N)$, in particular we obtain a Weyl and a Demazure character formula for the irreducible polynomial representations (Section 4, Theorem 4.5 and 4.11).

In a second time, we lift these action to an action of $H_n(q)$. This allows us to define a q-analogue of the Weyl symmetrizer (Section 5), from which we obtain a notion of quasi-symmetric Hall–Littlewood functions G_I analogous to (3), (Subsection 6.2). By duality, we obtain non-commutative Hall–Littlewood functions H_I analogous to Q'_μ (Subsection 6.3). We investigate the properties of these generalized Hall–Littlewood functions, in particular, we describe explicitly their expressions on the generalized Schur bases and their structure constants (Theorems 6.6, 6.13, and 6.15). This allows us to prove that the H_I behave like the Q'_μ under specialization of q at roots of unity (Corollary 6.19).

Such a property was expected from the results of [24, 25] in the commutative case, and from the analogs of Klyachko's idempotent in [11]. Indeed the non-commutative function

$$K_n(q) = \sum_{|I| = n} q^{\mathrm{Maj}(I)} R_I \tag{7}$$

has for commutative image the Hall–Littlewood function $\tilde{Q}'_{(1^n)}$, whose specialization at $q = e^{2\pi i/n}$ is the power sum p_n, but as an element of the descent algebra becomes

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{Maj}(\sigma)} \sigma \tag{8}$$

whose specialization $q = e^{2\pi i/n}$ is, up to a scalar factor, Klyachko's idempotent [19].

Our non-standard action of the Hecke algebra can be extended to the affine Hecke algebra $\tilde{H}(q)$. In this way, it is possible to characterize some two parameter bases of quasi-symmetric and non-commutative symmetric functions which diagonalize the center of $\tilde{H}(q)$. But this does not seem to provide the proper generalization of Macdonald's functions [15].

2. BACKGROUND

2.1. *Compositions*

A *composition* $K = (k_1, ..., k_p)$ is a p-tuple of positive integers. The integers k_i are called *the parts* of the composition. The number p is called the *length* of K and is denoted by $\ell(K)$. Let $n = |K| = k_1 + \cdots + k_p$ be the *sum* of the composition K. We say that K is a composition of n and we write it by $K \models n$. A weakly decreasing composition is called a *partition*. A p-tuple of non-negative integers of sum n is called a *pseudo composition*. Let $I = (i_1, ..., i_q)$ and $J = (j_1, ..., j_p)$ be two compositions. By $I \cdot J$ we mean the concatenation of the two compositions $I \cdot J = (i_1, ..., i_q, j_1, ..., j_p)$. We denote by $I \triangleright J$ the composition defined by $(i_1, ..., i_q + j_1, ..., j_p)$. Sometimes we need the composition of length r with all parts equal to i; it will be denoted by (i^r) or briefly by i^r.

Subsets of $\{1, ..., n-1\}$ are in one-to-one correspondence with compositions of n:

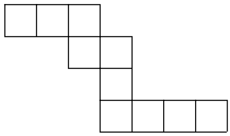
$$S = \{i_1 < i_2 < \cdots < i_p\} \mapsto \mathrm{C}(S) = (i_1, i_2 - i_1, i_3 - i_2, ..., n - i_p). \tag{9}$$

The inverse bijection (*descent set of a composition*) is given by

$$K=(k_1, ..., k_p) \mapsto \mathrm{Des}(K)=\{k_1+\cdots+k_j,\ j=1, ..., p-1\}. \tag{10}$$

For instance, the composition $(3, 1, 2, 1, 2, 2)$ of 11 corresponds to the subset $\{3, 4, 6, 7, 9\}$ of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

A composition can be represented by a skew Young diagram called a *ribbon diagram* of shape I (see [40]). For example, the ribbon diagram of the composition $I=(3, 2, 1, 4)$ is



The *conjugate* composition $I^{\sim}$ of I is obtained by reading from left to right the heights of the columns of the ribbon diagram of I. On the descent set, the conjugate is the complement in $\{1...n\}$. For example, the conjugate of $(3, 2, 1, 4)$ is $(1, 1, 2, 3, 1, 1, 1)$. These two compositions correspond to descent sets $\{3, 5, 6\}$ and $\{1, 2, 4, 7, 8, 9\}$.

Let I and J be two compositions of the same number n. We say that I is *finer* than J iff $\mathrm{Des}(I) \supseteq \mathrm{Des}(J)$. We will denote this by $I \succcurlyeq J$. This can be read on compositions in the following way: Let $J=(j_1, ..., j_p)$. The composition I is finer than J iff there exist compositions I_1 of j_1, I_2 of j_2,..., I_p of j_p such that $I=I_1 \cdot I_2 \cdots I_p$ is the composition obtained by concatenating $I_1, ..., I_p$ one after another. In this case we call the composition $\mathrm{Bre}(I, J)=(\ell(I_1), ..., \ell(I_p))$ the *refining composition.*

Keeping the previous notations, let us write I as the concatenation of the $I_j=(i^j_1, i^j_2, ..., i^j_{l_j})$ for $j=1\cdots p$. We get

$$I=(i^1_1, ..., i^1_{l_1}\cdot i^2_1, ..., i^2_{l_2}\cdot \cdots \cdot i^p_1, ..., i^p_{l_p}). \tag{11}$$

We regard I as a q-tuple with $q=l_1+l_2+\cdots+l_p$ in which some parts are separated by "," and some other by "$\cdot$". Then if we replace all subsequences between the "," by the sum of their parts minus the number of "$\cdot$", we get a composition of $n-\ell(J)+1$ which we denote by I/J. The composition I/J is the composition associated with the complement of the set $\mathrm{Des}(J)$ in $\mathrm{Des}(I)$, considered as a subset of the complement of $\mathrm{Des}(J)$ in $\{1, ..., n-1\}$. Let J be a fixed composition of n. The above algorithm defines an order preserving one-to-one correspondence between compositions finer than J and compositions of $n-\ell(J)+1$.

EXAMPLE 2.1. Let $I=(2, 2, 1, 2, 1, 1, 1, 2, 1, 1)$ and $J=(2, 3, 5, 2, 1, 1)$, thus we write

$$J = (2\,,\quad 3\quad,\qquad 5\qquad,2\,,1\,,1)$$

$$I = (2\cdot 2, 1\cdot 2, 1, 1, 1\cdot 2\cdot 1\cdot 1)$$

$$\mathrm{Bre}(I, J) = (1\,,\quad 2\quad,\qquad 4\qquad,1\,,1\,,1)$$

and thus

$$I = (2\cdot 2\,,1\cdot 2\,,1\,,1\,,1\cdot 2\cdot 1\cdot 1)$$

$$I/J = (\quad 3\quad,\quad 2\quad,1\,,1\,,\qquad 2\qquad).$$

On descent sets, $\mathrm{Des}(J) = \{2, 5, 10, 12, 13\} \subset \{1 < \cdots < 13\}$ and $\mathrm{Des}(I) = \{2, 4, 5, 7, 8, 9, 10, 12, 13\}$. Then $\mathrm{Des}(I)\backslash\mathrm{Des}(J) = \{4, 7, 8, 9\}$ considered as a subset of $\{1<3<4<6<7<8<9<11\}$ which is, after renumbering, $\{3, 5, 6, 7\}$ considered as a subset of $\{1 < \cdots < 8\}$ and so is associated to the composition $(3, 2, 1, 1, 2)$.

The following properties are immediate consequences of the definitions.

Proposition 2.2. *Let I and J be two compositions of the same number n. Suppose that $I \succcurlyeq J$. Then $J^{\sim} \succcurlyeq I^{\sim}$ and* $(\mathrm{Bre}(I, J))^{\sim} = J^{\sim}/I^{\sim}$.

Example 2.3. With the notations of the former example, we have

$$I^{\sim} = (1, 2, 3, 5, 3) \qquad \text{and} \qquad ({}^{\sim}J) = (1, 2, 1, 2, 1, 1, 1, 2, 3).$$

We verify that

$$J^{\sim}/I^{\sim} = (2, 2, 1, 1, 4) = (1, 2, 4, 1, 1, 1)^{\sim} = (\mathrm{Bre}(I, J))^{\sim}$$

$$\mathrm{Bre}(J^{\sim}, I^{\sim}) = (1, 1, 2, 4, 1) = (3, 2, 1, 1, 2)^{\sim} = (I/J)^{\sim}.$$

Finally let $K = (k_1, ..., k_p)$ be a composition of n. The *major index* of MacMahon [40] is defined to be

$$\mathrm{Maj}(K) = \sum_{i \in \mathrm{Des}(K)} i = (p-1)\, k_1 + (p-2)\, k_2 + \cdots + 2k_{p-2} + k_{p-1}. \tag{12}$$

It verifies the equation

$$\mathrm{Maj}(K) + \mathrm{Maj}(K^{\sim}) = \frac{n(n-1)}{2}. \tag{13}$$

For example, if $J = (2, 3, 5, 2, 1, 1)$, then $\mathrm{Maj}(J) = 2 + 5 + 10 + 12 + 13 = 42.$ and $\mathrm{Maj}(J^{\sim}) = 1 + 3 + 4 + 6 + 7 + 8 + 9 + 11 = 49$ and $42 + 49 = 91 = \frac{13\cdot 14}{2}$.

2.2. *Quasi-Symmetric Functions*

Let $X = \{x_1 < x_2 < \cdots < x_n\}$ denote a totally ordered set of commutative indeterminates. X is called the alphabet. By $\mathcal{P}(X)$ (resp. $\mathcal{P}_k(X)$), we mean the set of the subsets (resp. k-elements subsets) of the alphabet X.

Let m be the monomial $x_1^{m_1} \cdots x_n^{m_n}$ where the m_i are possibly zero. For readability, we identify m with the pseudo-composition $[m_1, m_2, ..., m_n]$. We define the *support* of m as the set $A \in \mathcal{P}(X)$ of the x_i whose exponent is non-zero and the composition K obtained by removing the zeros in the sequence $(m_1, m_2, ..., m_n)$. In the sequel we write A^K in place of the monomial m. For example, if $X = \{x_1 < x_2 < x_3 < x_4\}$, we write $x_1^2 x_3 = [2, 0, 1, 0] = \{x_1, x_3\}^{(2,1)}$ and $x_1^3 x_2^5 x_4 = [3, 5, 0, 1] = \{x_1, x_2, x_4\}^{(3,5,1)}$.

A polynomial $f \in \mathbb{C}[X]$ is said to be *quasi-symmetric* iff for each composition $I = (i_1, ..., i_r)$ the coefficient of the monomial A^I is independent of the set of variables $A \in \mathcal{P}_r(X)$. The quasi-symmetric polynomials form a subalgebra of $\mathbb{C}[X]$ denoted by $\mathbf{Qsym}_n$. It is often convenient to let $n \to \infty$ and to take the inverse limit in the category of graded ring. Hence we get an algebra called the algebra of *quasi-symmetric functions* [12]. Such functions can be seen as formal sums of monomials on an infinite alphabet $X = \{x_1 < x_2 < \cdots < x_n < \cdots\}$.

It is clear that the family of so-called *quasi-monomial functions* defined by

$$M_I = \sum_{A \in \mathcal{P}_r(X)} A^I = \sum_{j_1 < \cdots < j_r} x_{j_1}^{i_1} \cdots x_{j_r}^{i_r} = \sum_{k \to I} x^k \tag{14}$$

labeled by compositions $I = (i_1 ..., i_r)$ form a basis of $\mathbf{Qsym}$. The last sum is over all pseudo compositons $k \to I$ of length n obtained by inserting zeros in the composition I, noting $K \to I$ the suppression of zeros. For example,

$$M_{(2,1)} = \{x_1, x_2\}^{(2,1)} + \{x_1, x_3\}^{(2,1)} + \{x_1, x_4\}^{(2,1)} + \{x_2, x_3\}^{(2,1)} + \{x_2, x_4\}^{(2,1)} + \{x_3, x_4\}^{(2,1)}$$

and for readability, we prefer to write

$$M_{(2,1)} = [2, 1, 0, 0] + [2, 0, 1, 0] + [2, 0, 0, 1] + [0, 2, 1, 0] + [0, 2, 0, 1] + [0, 0, 2, 1]$$

instead of $M_{(2,1)} = x_1^2 x_2 + x_1^2 x_3 + x_1^2 x_4 + x_2^2 x_3 + x_2^2 x_4 + x_3^2 x_4$.

Another important basis of $\mathbf{Qsym}$ is given by the *quasi-ribbon functions*. In [12], Gessel defined them as characteristic functions of permutations with given descent set. They appear to play the same role as the Schur functions in the theory of symmetric function [21, 22]. One can define them by

$$F_I = \sum_{I \leqslant J} M_J, \tag{15}$$

e.g., $F_{122} = M_{122} + M_{1112} + M_{1211} + M_{11111}$. It is important to note that F_I is the commutative image of the sum of all quasi-ribbon words of shape I ([12], see also Section 4). Their characteristic properties are now explained by the fact that the algebra of quasi-symmetric functions is in natural duality with the Hopf algebra of noncommutative symmetric functions, which is nothing but the direct sum of the descent algebras of all symmetric groups [41].

2.3. *Noncommutative Symmetric Functions*

The algebra of *noncommutative symmetric functions* [11] is the free associative algebra $\mathbf{Sym} = \mathbb{C}\langle S_1, S_2, ...\rangle$ generated by an infinite sequence of noncommutative indeterminates S_k, called *complete* symmetric functions. For a composition $I = (i_1, i_2, ..., i_r)$, one sets $S^I = S_{i_1} S_{i_2} \cdots S_{i_r}$.

The family (S^I) is a linear basis of **Sym**. A useful realization can be obtained by taking an infinite alphabet $A = \{a_1, a_2, ...\}$ and defining its complete homogeneous symmetric functions by the generating function

$$\sum_{n \geqslant 0} t^n S_n(A) = (1 - t\, a_1)^{-1} (1 - t\, a_2)^{-1} (1 - t\, a_3)^{-1} \cdots. \tag{16}$$

Then $S_n(A)$ appears as the sum of all nondecreasing words of length n. Note that these functions are not symmetric in the usual sense. These are invariant for a more subtle action due to Lascoux and Schützenberger [29] (see also [11]). The role of Schur functions is played by the noncommutative *ribbon Schur functions* R_I defined by

$$R_I = \sum_{J \leqslant I} (-1)^{\ell(I) - \ell(J)} S^J. \tag{17}$$

The family (R_I) form a basis of **Sym**. In the realization of **Sym** given by Eq. (16), R_I reduces to the sum of all words of shape I [11].

The pairing $\langle \cdot, \cdot \rangle$ between **Qsym** and **Sym** is defined by $\langle M_I S^J \rangle = \delta_{IJ}$ or equivalently $\langle F_I, R_J \rangle = \delta_{IJ}$ (cf. [41, 11]). This duality can be interpreted as the canonical duality between the Grothendieck groups respectively associated with finite dimensional and projective modules over 0-Hecke algebras ([8, 20], see also Section 4).

3. QUASI-SYMMETRIZING ACTIONS

Let $\mathfrak{S}_n$ denote the nth symmetric group. For $i=1, 2, ..., n-1$, let σ_i denotes the elementary transposition $(i, i+1)$. It is a well known fact that $\mathfrak{S}_n$ is generated by the $(\sigma_i)_{i=1, ..., n-1}$ with the presentation

$$\begin{aligned} \sigma_i^2 &= 1 && \text{for} \quad 1 \leqslant i \leqslant n-1, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i && \text{for} \quad |i-j| > 1, \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} && \text{for} \quad 1 \leqslant i \leqslant n-2. \end{aligned} \tag{18}$$

These relations are called Moore–Coxeter relations. The last two relations are called braid relations. A *reduced word* $\sigma = \sigma_{i_1} \cdots \sigma_{i_p}$ for the permutation σ is a minimal length decomposition. The length $\ell(\sigma) = p$ of such a decomposition is called the *length of the permutation* σ. It is equal to the number of inversions of σ. The number $\varepsilon(\sigma) = (-1)^{\ell(\sigma)}$ is called the sign of σ.

The permutation of minimal length is the identity Id. The permutation of maximal length is the permutation $n, n-1, ..., 1$ (such that i is sent to $n+1-i$). We will denote this permutation by ω. It is of length $n(n-1)/2$ and it has many reduced decompositions, among them

$$\omega = \sigma_1(\sigma_2\sigma_1)(\sigma_3\sigma_2\sigma_1) \cdots (\sigma_{n-1} \cdots \sigma_2\sigma_1) \tag{19}$$

$$= (\sigma_1\sigma_2 \cdots \sigma_{n-1}) \cdots (\sigma_1\sigma_2\sigma_3)(\sigma_1\sigma_2)\, \sigma_1. \tag{20}$$

DEFINITION 3.1. A permutation σ admits a *descent* at position i if $\sigma(i) > \sigma(i+1)$ and admits a *rise* at position i otherwise.

This is equivalent to the fact that there exists a reduced word for σ ending with σ_i. Therefore this notion makes sense for arbitrary Coxeter groups, in this context one would rather say that σ_i is a descent for σ. Another way to express this definition is the following: an elementary transposition σ_i is a descent for σ if and only if there exists a permutation σ' such that $\sigma = \sigma' \sigma_i$ and $\ell(\sigma) = \ell(\sigma') + 1$.

DEFINITION 3.2. Symmetrically, one says that the permutation σ admits a *recoil* at i if $\sigma^{-1}(i) > \sigma^{-1}(i+1)$.

For all $i=1, ..., n-1$, there is a descent and a recoil at position i in the maximal permutation ω. This means that there exists a reduced word for the maximal permutation starting or ending with any elementary transposition.

3.1. *The Action of the Symmetric Group*

This section is devoted to the study of a new action of the symmetric group on polynomials whose invariants are the quasi-symmetric polynomials. After defining this action, we give some character formulas. In particular, we show that there are only very few irreducible representations occurring in it.

As usual permutations act on polynomials by the formula

$$(\sigma f)(x_1, ..., x_n) = f(x_{\sigma(1)}, ..., x_{\sigma(n)}) \tag{21}$$

or with the notations of the previous sections,

$$\sigma[m_1, ..., m_n] = [m_{\sigma^{-1}(1)}, ..., m_{\sigma^{-1}(n)}]. \tag{22}$$

DEFINITION 3.3. Let $m = x_1^{k_1} \cdots x_n^{k_n} = [k_1, ..., k_n]$ be a monomial. The operator $\boldsymbol{\sigma}_i$ acts on m by

$$\begin{aligned} &\boldsymbol{\sigma}_i[k_1, ..., k_i, k_{i+1}, ..., k_n] \\ &\quad = \begin{cases} [k_1, ..., k_{i+1}, k_i, ..., k_n] & \text{if} \quad k_i = 0 \quad \text{or} \quad k_{i+1} = 0 \\ [k_1, ..., k_i, k_{i+1}, ..., k_n] & \text{if} \quad k_i \neq 0 \quad \text{and} \quad k_{i+1} \neq 0. \end{cases} \end{aligned} \tag{23}$$

The map $\sigma_i \to \boldsymbol{\sigma}_i$ defines a faithful action of the symmetric group $\mathfrak{S}_n$ on $\mathbb{C}[X]$.

To see this we recast (23) in the following form:

PROPOSITION 3.4. *Let $m = [k_1, ..., k_n] = A^I$ be a monomial. The quasi-symmetrizing action of a permutation $\sigma \in \mathfrak{S}_n$ is given by the operator $\boldsymbol{\sigma}$,*

$$\boldsymbol{\sigma}(A^I) = (\sigma A)^I = \{x_{\sigma(i)} \mid x_i \in A\}^I. \tag{24}$$

Note 3.5. Recall that $A \subset X$ inherit a total order from X. The reader has to take care to the fact that to get the monomial associated with A^I on has to write the elements of A in *increasing order*. For example, let σ be the permutation which exchanges 1 and 4, and fixes 2 and 3. Then

$$\sigma\{x_1, x_3\} = \{x_{\sigma(1)}, x_{\sigma(3)}\} = \{x_4, x_3\} = \{x_3 < x_4\}$$

and thus if $\boldsymbol{\sigma}$ denotes the associated quasi-symmetrizing operator, then

$$\boldsymbol{\sigma}[1, 0, 2, 0] = (\sigma\{x_1, x_3\})^{(1,2)} = \{x_3, x_4\}^{(1,2)} = [0, 0, 1, 2],$$

so that $\boldsymbol{\sigma}(x_1 x_3^2) = x_3 x_4^2$.

Proof. It is easy to see that both expressions give the same operator when $\sigma=\sigma_i$. For example, on $[k_1, k_2, 0]=\{x_1, x_2\}^{(k_1, k_2)}$, the transposition σ_1 gives $[k_1, k_2, 0]$, which is $(\sigma_1\{x_1, x_2\})^{(k_1, k_2)}$ and σ_2 gives $[k_1, 0, k_2]$ which is $(\sigma_2\{x_1, x_2\})^{(k_1, k_2)}=(\{x_1, x_3\})^{(k_1, k_2)}$. Since it is obvious that the expression (24) defines an action of the symmetric group, the operators defined by (23) verify the defining relations of the symmetric group. So, both expressions define the same action of $\mathfrak{S}_n$. Moreover, this action is clearly faithful, since the action on monomials of degree 1, that is the variables, allows us to reconstruct the permutation. ▮

EXAMPLE 3.6. From Eq. (24), we get that

$$\boldsymbol{\sigma}_1(x_1^6x_2^2)=\boldsymbol{\sigma}_1[6, 2, 0]=(\sigma_1\{x_1, x_2\})^{(6, 2)}=\{x_1, x_2\}^{(6, 2)}=x_1^6x_2^2$$
$$\boldsymbol{\sigma}_1(x_1^6x_3^2)=\boldsymbol{\sigma}_1[6, 0, 2]=(\sigma_1\{x_1, x_3\})^{(6, 2)}=\{x_2, x_3\}^{(6, 2)}=x_2^6x_3^2$$

which agrees with (23).

Note 3.7. In the sequel, we will see that the quasi-symmetrizing action is not a faithful action of the *algebra* of the symmetric group. Thus we have to distinguish between abstract permutations and quasi-symmetrizing operators acting on polynomials. We use the following convention: The abstract elements of the symmetric group or Hecke algebra will be denoted by normal-type letter for example σ or T. The associated quasi-symmetrizing operators will appear in bold-type such as $\boldsymbol{\sigma}$ or $\mathbf{T}$. Moreover since the classical actions are faithful, we don't need to distinguish if from abstract permutations, so we keep the normal-type for it.

Note 3.8. It is important to see that the quasi-symmetrizing action is an action on the vector space of polynomials, with no relation with the multiplicative structure: $(\boldsymbol{\sigma}_1x_1^2)(\boldsymbol{\sigma}_1x_2)=x_1x_2^2$ whereas $\boldsymbol{\sigma}_1(x_1^2\,x_2)=x_1^2\,x_2$. In particular, though symmetric functions are scalars for the classical σ_i, in our case, the only polynomials f such that $\boldsymbol{\sigma}_i(fg)=f\boldsymbol{\sigma}_i(g)$ for all g are the constant ones.

Let us have a closer look at this representation of the symmetric group. It is possible to give the character of this representation. Our notations for symmetric functions are those of [38].

Recall that the direct sum $\bigoplus_{n\geqslant 0} R(\mathfrak{S}_n)$ of Grothendieck rings of all symmetric groups is in natural isomorphism with the ring of symmetric functions, by the so-called Frobenius characteristic map *ch*. It sends the irreducible character χ^λ to the Schur function s_λ. The product of symmetric functions corresponds to induction from $\mathfrak{S}_n\times\mathfrak{S}_p$ to $\mathfrak{S}_{n+p}$. The quasi-symmetrizing action is compatible with the usual grading of polynomial rings. Hence, one can define the graded characteristic ch_t of the representation on

$\mathbb{C}[X]$ as the generating series of the characteristics of the representations on the homogeneous components $\mathbb{C}_i[X]$,

$$\mathrm{ch}_t(\mathbb{C}[X]) = \sum_{i=0}^{\infty} \mathrm{ch}(\mathbb{C}_i[X])\, t^i. \tag{25}$$

PROPOSITION 3.9. *The graded characteristic of the quasi-symmetrizing action of $\mathfrak{S}_n$ over the space of polynomials is given by*

$$\mathrm{ch}_t(\mathbb{C}[x_1, ..., x_n]) = \sum_{m=0}^{n} \frac{t^m}{(1-t)^m} h_{(m,\, n-m)}. \tag{26}$$

Proof. This proposition relies on the following well known lemma

LEMMA 3.10. *Let $m \leqslant \frac{n}{2}$. Consider $\mathfrak{S}_n$ acting on $\mathscr{P}_m(X)$. This representation is isomorphic to the induction of the trivial representation of the Young subgroup $\mathfrak{S}_m \times \mathfrak{S}_{n-m}$ to $\mathfrak{S}_n$.*

The character formula for representations induced from Young subgroups is well known (see, e.g., [38, Section I.7, pp. 112–114]). For $m \leqslant \frac{n}{2}$ it gives

$$\mathrm{ch}(\mathscr{P}_m(X)) = \mathrm{ch}(\mathfrak{S}_n/(\mathfrak{S}_m \times \mathfrak{S}_{n-m})) = h_{(n-m,\, m)} = \sum_{k \leqslant m} s_{(n-k,\, k)}.$$

Note that the first two equalities are still true even if $m > \frac{n}{2}$, since $h_{(n-m,\, m)} = h_{(m,\, n-m)}$. The representation $\mathscr{P}_m(X)$ occur as many times as the number of compositions of length m. The generating series of such compositions counted by their sum is

$$\sum_{i=1}^{\infty} \left(\sum_{K \models i,\, \ell(K) = m} t^i \right) = \frac{t^m}{(1-t)^m}. \tag{27}$$

It remains to see that the representation on the constant polynomials is the trivial one. Its character is $s_n = h_n$. This corresponds to the case $m = 0$. ∎

COROLLARY 3.11. *The only irreducible representations occurring in $\mathbb{C}[X]$ under the quasi-symmetrizing action are the trivial one $\chi^{(n)}$ and the two parts representations $\chi^{(n-m,\, m)}$ for $m \leqslant \frac{n}{2}$.*

The alternating sum of the permutations of a Young subgroup corresponding to a partition λ kills all irreducible representation indexed by partition μ such that the conjugate of λ is not finer than μ. In our case, taking a subgroup

$$\mathfrak{S}_1 \times \cdots \mathfrak{S}_1 \times \mathfrak{S}_3 \times \mathfrak{S}_1 \times \cdots \times \mathfrak{S}_1, \tag{28}$$

one gets the equality

$$\boldsymbol{\sigma}_i \boldsymbol{\sigma}_{i+1} \boldsymbol{\sigma}_i - \boldsymbol{\sigma}_{i+1} \boldsymbol{\sigma}_i - \boldsymbol{\sigma}_i \boldsymbol{\sigma}_{i+1} + \boldsymbol{\sigma}_i + \boldsymbol{\sigma}_{i+1} - 1 = 0. \tag{29}$$

It allows us to express the action of every permutation as a linear combination of the action of permutations such that no reduced word has a sequence $\sigma_i \sigma_{i+1} \sigma_i$. These are exactly the permutations that avoid the pattern 321. Recall that a permutation avoids the pattern 321 if there does not exist $i < j < k$ such that $\sigma(i) > \sigma(j) > \sigma(k)$. These are the permutations whose insertion tableau under the Robinson–Schensted algorithm has at most 2 rows.

As a consequence one has:

THEOREM 3.12. *The image of* $\mathbb{C}[\mathfrak{S}_n]$ *in* $\mathrm{End}(\mathbb{C}[X])$ *is the quotient of* $\mathbb{C}[\mathfrak{S}_n]$ *by the ideal generated by*

$$(\sigma_i \sigma_{i+1} \sigma_i - \sigma_{i+1} \sigma_i - \sigma_i \sigma_{i+1} + \sigma_i + \sigma_{i+1} - 1)_{i=1 \cdots n-2}. \tag{30}$$

The family $(\boldsymbol{\sigma})_\sigma$ *where* σ *goes along the set of all permutations avoiding the pattern* 321 *is a basis of the image of* $\mathbb{C}\langle \mathfrak{S}_n \rangle$ *in* $\mathrm{End}(\mathbb{C}[X])$.

Note that the element (30) admits the following factorization due to Young,

$$(\sigma_i - 1)(\sigma_{i+1} \sigma_i - \sigma_{i+1} + 1). \tag{31}$$

We will give another proof of this theorem in the case of the generic Hecke algebra (see Theorem 5.10).

Note. 3.13. Let A be a p-subset of $\{1, ..., n\}$. There are exactly $p!\,(n-p)!$ permutations fixing m. Thus, for each B of cardinal p, there are exactly $p!(n-p)!$ permutations σ such that $\sigma A = B$. Among them there is a shortest one. Is it easy to see that the set of such permutations is the set of permutations avoiding the pattern 321.

Since the cosets $\mathfrak{S}_n / (\mathfrak{S}_m \times \mathfrak{S}_{n-m})$ play an important role in the sequel, let us record some of their properties.

PROPOSITION 3.14. *Let* $A = \{1, ..., p\}$.

- *The image of* A *under* σ *depends only on the class of* σ *in* $\mathfrak{S}_n/(\mathfrak{S}_p \times \mathfrak{S}_{n-p})$. *This defines a bijection between* p*-subsets of* $\{1, ..., n\}$ *and cosets in* $\mathfrak{S}_n/(\mathfrak{S}_p \times \mathfrak{S}_{n-p})$.
- *Each coset* c *in* $\mathfrak{S}_n/(\mathfrak{S}_p \times \mathfrak{S}_{n-p})$ *contains a unique permutation* σ_c *of minimum length.*
- *The permutations* σ_c *are exactly the permutations occurring in the shuffle*

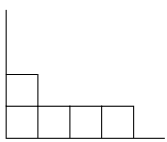
$$12 \cdots p \sqcup\!\sqcup p+1 \cdots n.$$

We denote this set by $\mathfrak{S}_{n/p}$.

There are also the two useful encodings for a coset: one can replace the p-subset A by a word w of weight $1^p 0^{n-p}$, putting a $w_i = 1$ if $i \in A$. Another way is given by partition λ fitting in $(n-p)^p$. Reading the upper right border of the Ferrers diagram (in French notations) as a word on the letter "south" or "east,"

Classe of 124835679 mod $\mathfrak{S}_4 \times \mathfrak{S}_5$

$$\leftrightarrow \{1, 2, 4, 8\} \leftrightarrow (110100010) \leftrightarrow SSESEEESE$$

$\leftrightarrow$  .

The main property of the quasi-symmetrizing action is the following:

PROPOSITION 3.15. *A polynomial* $f \in \mathbb{C}[x_1 < \cdots < x_n]$ *is quasi-symmetric if and only if* $\boldsymbol{\sigma} f = f$ *for all permutations* $\sigma \in \mathfrak{S}_n$.

Proof. By definition, a polynomial is quasi-symmetric iff the coefficient of A^K is independent of $A \in \mathcal{P}_p(X)$. But the symmetric group acts transitively on $\mathcal{P}_p(X)$. Hence, if A and B are two p-subsets of X, there is a permutation σ such that $\sigma A = B$ and therefore $\boldsymbol{\sigma} A^I = B^I$. ∎

Is is easy to see that the quasi-monomial polynomial M_K is the sum of the orbit of X^K under the quasi-symmetrizing action. Moreover we can get it in the following way:

PROPOSITION 3.16. *Let $K=(k_1, ..., k_p)$ be a composition. Let $m=X^K=[k_1, ..., k_p, 0, ..., 0]$. Then*

$$M_K=\frac{1}{p!\,(n-p)!}\sum_{\sigma\in\mathfrak{S}_n}\boldsymbol{\sigma}m=\sum_{\sigma\in\mathfrak{S}_n/(\mathfrak{S}_p\times\mathfrak{S}_{n-p})}\boldsymbol{\sigma}m. \tag{32}$$

Proof. It is sufficient to see that $p!\,(n-p)$ is the number of permutations which fix the monomial $[k_1, ..., k_p, 0, ..., 0]$. This is the size of a coset of $\mathfrak{S}_n/(\mathfrak{S}_p\times\mathfrak{S}_{n-p})$. ▮

3.2. *Hecke Algebra, Quasi-Symmetrizing Divided Differences*

Let q be a formal or complex parameter. The Hecke algebra $H_n(q)$ of the symmetric group $\mathfrak{S}_n$ (type A_{n-1}) is the $\mathbb{Q}[q, q^{-1}]$ algebra generated by elements $(T_i)_{i=1, ..., n-1}$ with the relations

$$\begin{aligned} T_i^2&=(q-1)\,T_i+q &&\text{for}\quad 1\leqslant i\leqslant n-1,\\ T_iT_j&=T_jT_i &&\text{for}\quad |i-j|>1,\\ T_iT_{i+1}T_i&=T_{i+1}T_iT_{i+1} &&\text{for}\quad 1\leqslant i\leqslant n-2.\end{aligned} \tag{33}$$

For generic q (different from 0 or a non-trivial root of unity) $H_n(q)$ is isomorphic to $\mathbb{Q}[\mathfrak{S}_n]$.

Let $\sigma=\sigma_{i_1}\cdots\sigma_{i_p}$ be a reduced word. The defining relations of $H_n(q)$ ensure that the element $T_{i_1}\cdots T_{i_p}$ is independent of the chosen reduced word for σ. We denote this element by T_σ. By convention, $T_{\mathrm{Id}}=1$, where Id is the identity of the symmetric group. The family $(T_\sigma)_{\sigma\in\mathfrak{S}_n}$ is a basis of the Hecke algebra.

With this choice for the quadratic relation, it is possible to put $q=0$ in the definition of the Hecke Algebra. We obtain the degenerate Hecke algebra $H_n(0)$. The relations become

$$\begin{aligned} T_i^2&=T_i &&\text{for}\quad 1\leqslant i\leqslant n-1,\\ T_iT_j&=T_jT_i &&\text{for}\quad |i-j|>1,\\ T_iT_{i+1}T_i&=T_{i+1}T_iT_{i+1} &&\text{for}\quad 1\leqslant i\leqslant n-2.\end{aligned} \tag{34}$$

The goal of the remainder of this section is to show that there exists an action of the degenerate Hecke algebra on polynomials which fixes the quasi-symmetric polynomials. On the way, we get a new expression for the quasi-ribbon polynomials. The method is similar to the one of Lascoux and Schützenberger. In [32], these authors described several families of operators acting on $\mathbb{C}[X]$ which satisfy the braid and Hecke relations. In particular the so-called isobaric divided differences, given by

$$\pi_i f = (1+\sigma_i)\frac{(x_i f)}{x_i - x_{i+1}}$$

define an action of the degenerate Hecke algebra $H_n(0)$

Definition 3.17. Let f be a polynomial and $i<n$. The quasi-symmetrizing isobaric divided differences are defined by

$$\boldsymbol{\pi}_i f = \frac{x_i f - x_{i+1}\boldsymbol{\sigma}_i f}{x_i - x_{i+1}} \qquad \text{and} \qquad \bar{\boldsymbol{\pi}}_i = \boldsymbol{\pi}_i - \mathrm{Id}. \tag{35}$$

Proposition 3.18. *The quasi-symmetrizing operator σ_i is characterized by the fact that*

$$\boldsymbol{\sigma}_i f = f \Leftrightarrow \boldsymbol{\pi}_i f = f \tag{36}$$

and

$$\boldsymbol{\pi}_i(x_i^n) = \sum_{u+v=i} x_i^u x_{i+1}^v. \tag{37}$$

For example, $\boldsymbol{\pi}_1[1,2,3] = [1,2,3]$ and $\boldsymbol{\pi}_2[1,0,3] = -[1,2,1] - [1,1,2]$. The proposition is a consequence of the following easy lemma:

Lemma 3.19. *One has*

$$\boldsymbol{\sigma}_i\boldsymbol{\pi}_i = \boldsymbol{\pi}_i \qquad \textit{and} \qquad \boldsymbol{\sigma}_i\bar{\boldsymbol{\pi}}_i = \boldsymbol{\pi}_i - \boldsymbol{\sigma}_i \tag{38}$$

$$\bar{\boldsymbol{\pi}}_i\boldsymbol{\sigma}_i = -\bar{\boldsymbol{\pi}}_i \qquad \textit{and} \qquad \boldsymbol{\pi}_i\boldsymbol{\sigma}_i = -\bar{\boldsymbol{\pi}}_i + \boldsymbol{\sigma}_i, \tag{39}$$

where fg is the composition $f\circ g$ of the two operators f and g (the operators act on their right). We will still use this notation in the sequel.

This is useful in the proof of the following theorem:

Theorem 3.20. *The $\boldsymbol{\pi}_i$ operators satisfy*

$$\begin{aligned} \boldsymbol{\pi}_i^2 &= \boldsymbol{\pi}_i && \textit{for} \quad 1 \leqslant i \leqslant n-1, \\ \boldsymbol{\pi}_i\boldsymbol{\pi}_j &= \boldsymbol{\pi}_j\boldsymbol{\pi}_i && \textit{for} \quad |i-j|>1, \\ \boldsymbol{\pi}_i\boldsymbol{\pi}_{i+1}\boldsymbol{\pi}_i &= \boldsymbol{\pi}_{i+1}\boldsymbol{\pi}_i\boldsymbol{\pi}_{i+1} && \textit{for} \quad 1 \leqslant i \leqslant n-2. \end{aligned} \tag{40}$$

The $\bar{\pi}_i$ *operators satisfy*

$$\begin{aligned} \bar{\pi}_i^2 &= -\bar{\pi}_i && \text{for} \quad 1 \leqslant i \leqslant n-1, \\ \bar{\pi}_i \bar{\pi}_j &= \bar{\pi}_j \bar{\pi}_i && \text{for} \quad |i-j| > 1, \\ \bar{\pi}_i \bar{\pi}_{i+1} \bar{\pi}_i &= \bar{\pi}_{i+1} \bar{\pi}_i \bar{\pi}_{i+1} && \text{for} \quad 1 \leqslant i \leqslant n-2. \end{aligned} \tag{41}$$

Before proving this theorem, we give a more concise formulation. Let $\sigma = \sigma_{i_1}, ..., \sigma_{i_p}$ be a reduced word. The braid relations ensure that the operators $\pi_\sigma = \pi_{i_1} \cdots \pi_{i_p}$ and $\bar{\pi}_\sigma = \bar{\pi}_{i_1} \cdots \bar{\pi}_{i_p}$ are independent of the reduced word for σ. As usual, we set that $\pi_{\mathrm{Id}} = \mathrm{Id}_{\mathbb{C}[X]}$.

COROLLARY 3.21. *The mappings* $T_\sigma \mapsto (-1)^{\ell(\sigma)} \pi_\sigma$ *and* $T_\sigma \mapsto \bar{\pi}_\sigma$ *defines two actions of the degenerate Hecke algebra.*

The classical theory of the Hecke algebra says that if one of these two families of operators satisfies the Hecke relations then the other satisfies it again. Indeed π and $\bar{\pi}$ are the image one of each other under the classical involution of the degenerate Hecke algebra. This involution corresponds to the exchange of the two roots $q_1 = 0$ and $q_2 = 1$ in the quadratic Hecke equation

$$(T_i - q_1)(T_i - q_2) = 0. \tag{42}$$

Thus we only need to prove the formulas for $\bar{\pi}$.

Proof. Since $\bar{\pi}_i$ acts only on x_i, x_{i+1}, the proof of the theorem reduce to the case $n = 2$ and $n = 3$.

In the case $n = 2$, given and f, on has $\pi_1 f = \boldsymbol{\sigma}_i \pi_1 f$ and therefore $\pi_1^2 f = \pi_1 f$ and $\bar{\pi}_1^2 f = -\bar{\pi} f$.

In the case $n = 3$ the case to consider are $[a, 0, 0]$, $[0, a, 0]$, $[a, b, 0]$, $[a, 0, b]$. Let us just detail the example of $[0, a, 0]$. Let us consider the generating function

$$\sum_{a \geqslant 0} [0, a, 0] = \sum_{a \geqslant 0} x_2^a = \frac{1}{1 - x_2}. \tag{43}$$

Then one has

$$\frac{1}{1 - x_2} \xrightarrow{\bar{\pi}_1} \frac{-x_1}{(1 - x_2)(1 - x_2)} \xrightarrow{\bar{\pi}_2} \frac{x_3}{(1 - x_1)(1 - x_2)(1 - x_3)}$$
$$\xrightarrow{\bar{\pi}_2} 0.$$

The first two steps coïncide with the usual action (only one of the two variable x_i or x_{i+1} is present in the functions on which acts $\bar{\pi}_i$). The last step on the right gives 0 because the functions is symmetric in x_1 and x_2. Likewise one has

$$\frac{1}{1-x_2} \xrightarrow{\bar{\pi}_2} \frac{-x_3}{(1-x_2)(1-x_3)} \xrightarrow{\bar{\pi}_1} \frac{-x_2 x_3}{(1-x_1)(1-x_2)(1-x_3)} \xrightarrow{\bar{\pi}_2} 0.$$

The computations in the remaining cases are very similar. ∎

3.3. *Maximal Symmetrizer, Quasi-Ribbon polynomials*

Let ω be the maximal permutation $n, n-1, ..., 1$. The operator

$$\boldsymbol{\pi}_\omega = \sum_{\sigma \in \mathfrak{S}_n} \bar{\boldsymbol{\pi}}_\sigma \tag{44}$$

is called the *maximal quasi-symmetrizer*. It has this property:

Proposition 3.22. *$\boldsymbol{\pi}_\omega$ is a projector whose image is the space of quasi-symmetric polynomial.*

Proof. First, if f is quasi symmetric, $\boldsymbol{\pi}_i f = f$, for all $i \in \{1, ..., n-1\}$. It follows that $\boldsymbol{\pi}_\sigma f = f$ for all σ and in particular for ω.

Conversely, we have to show that $\boldsymbol{\pi}_\omega f$ is quasi-symmetric for all f. But, for any i, there exists a permutation σ' such that $\boldsymbol{\pi}_\omega = \boldsymbol{\pi}_i \boldsymbol{\pi}_{\sigma'}$ and hence by formula (38)

$$\boldsymbol{\sigma}_i \boldsymbol{\pi}_\omega = \boldsymbol{\sigma}_i \boldsymbol{\pi}_i \boldsymbol{\pi}_{\sigma'} = \boldsymbol{\pi}_i \boldsymbol{\pi}_{\sigma'} = \boldsymbol{\pi}_\omega.$$

This proves that $\boldsymbol{\pi}_\omega f$ is invariant by any elementary transposition and consequently by any permutation. By Proposition 3.15 it is quasi-symmetric. ∎

Corollary 3.23. *Let f be a polynomial. The following three properties are equivalent*:

(i) *f is quasi-symmetric,*

(ii) *$\boldsymbol{\pi}_\sigma f = f$ for all $\sigma \in \mathfrak{S}_n$.*

(iii) *$\boldsymbol{\pi}_\omega f = f$.*

Now, we want to express the image of a monomial by the maximal symmetrizer. If $K = (k_1, ..., k_p)$ is a composition, we say that the monomial $X^K = [x_1, ..., x_p, 0, ..., 0]$ is *dominant*. The following formula is of great interest:

THEOREM 24. *Let K be a composition and $M = X^K = [x_1, ..., x_p, 0, ..., 0]$ its associated dominant monomial. Its image under the total symmetrizer is the quasi-ribbon function*

$$\pi_\omega X^K = F_K. \tag{45}$$

We will not prove this theorem now, since there is a stronger statement in Theorem 3.29.

In the case where the monomial m is not dominant, the straightening rule is given by the fact that $\pi_\omega = \pi_\omega \pi_i$, for any i.

PROPOSITION 3.25 (Straightening Rule for Quasi-Ribbon Functions). *Let $m = [k_1, ..., k_n]$ be a monomial such that $k_p = 0$. for some $p < n$. Then*

$$\pi_\omega m = \begin{cases} -\sum_{\substack{u+v=k_{p+1} \\ u \neq 0,\, v \neq 0}} \pi_\omega [k_1, ..., u, v, ..., k_n] & \textit{if} \quad k_{p+1} > 1, \\ 0 & \textit{if} \quad k_{p+1} = 1, \\ m & \textit{otherwise.} \end{cases} \tag{46}$$

Proof. Since $\omega^{-1} = \omega$, a reduced word for the maximal permutation can end with any permutation. Therefore, for any i, there exists a σ' such that $\pi_\omega = \pi_{\sigma'} \pi_i$. Then $\pi_\omega \pi_i = \pi_{\sigma'} \pi_i^2 = \pi_\omega$, because $\pi_i^2 = \pi_i$. The end of the proof comes from formula (37). ∎

By an easy induction, we get the final formula:

PROPOSITION 3.26. *Suppose that $m = [k_1, ..., k_p, 0, ..., 0]$ is a monomial such that $k_p \neq 0$. Then*

$$\pi_\omega m = (-1)^{C_m} \sum_J F_J, \tag{47}$$

where C_m is the number of zero parts on the left of k_p. The sum is extended to all compositions obtained from m, by replacing maximal blocks of zeros followed by a non-zero part by a composition of this part. If such a composition does not exist, i.e. if the number of zeros is greater than the part, then the sum is null.

For example, let us compute $\pi_\omega [0, 0, 5, 0, 2, 2, 0]$. There are three zeroes before the last non-zero part, so that $C_m = -1$. The block 0, 0, 5 has to be replaced by a composition of 5 of length 3, the possible choice are (1, 1, 3), (1, 2, 2), (1, 3, 1), (2, 1, 2), (2, 2, 1), (3, 1, 1). The block 0, 2 can only be replaced by 1, 1. Finally we get that

$$\pi_\omega[0, 0, 5, 0, 2, 2, 0] = -(F_{(1,1,3,1,1,2)} + F_{(1,2,2,1,1,2)} + F_{(1,3,1,1,1,2)} + F_{(2,1,2,1,1,2)} + F_{(2,2,1,1,1,2)} + F_{(3,1,1,1,1,2)}).$$

Also, $\pi_\omega[0, 0, 2] = 0$ since it is not possible to break 2 into three non-zero parts.

Note that this can we viewed as a definition for F_m where m is not a dominant monomial.

3.4. *Partial Symmetrizers*

In this subsection, we present an explicit formula for the partial symmetrization of a dominant monomial. In particular this will prove Theorem 3.24. Before stating the formula, we need some combinatorial definitions.

Definition 3.27. We say that the monomial $m = A^I$ fits in a composition K if the composition I is finer than K.

This is equivalent to the fact that each partial sum of K appears as a partial sum of m,

$$\text{for all } i \leqslant p, \qquad \text{there exists a} \quad j \leqslant n \quad \text{such that} \quad \sum_{l \leqslant i} k_l = \sum_{l \leqslant j} m_l. \tag{48}$$

For example, $[1, 0, 2, 0, 2]$ fits in $(3, 2)$ but not in $(2, 1, 2)$. The quasi-ribbon function F_K is the sum of all monomials fitting in K.

Definition 3.28. Let $m = [m_1, ..., m_n]$ and $m' = [m'_1, ..., m'_n]$ be two monomials. We say that m is bigger than m' and we write $m \gg m'$, if for all $i < n$ one has

$$\sum_{j \leqslant i} m_j \geqslant \sum_{j \leqslant i} m'_j. \tag{49}$$

This defines a partial ordering on monomials.

Note that this order extends the natural ordering of partitions. Let us look at the restriction of this order on the monomials fitting in K. The greatest monomial is $X^K = [x_1, ..., x_p, 0, ..., 0]$, the smallest monomial is $[0, ..., 0, x_1, ..., x_p]$. The quasi-crystal graph [22] (see Figs. 1, 2) shows the graph of this ordering.

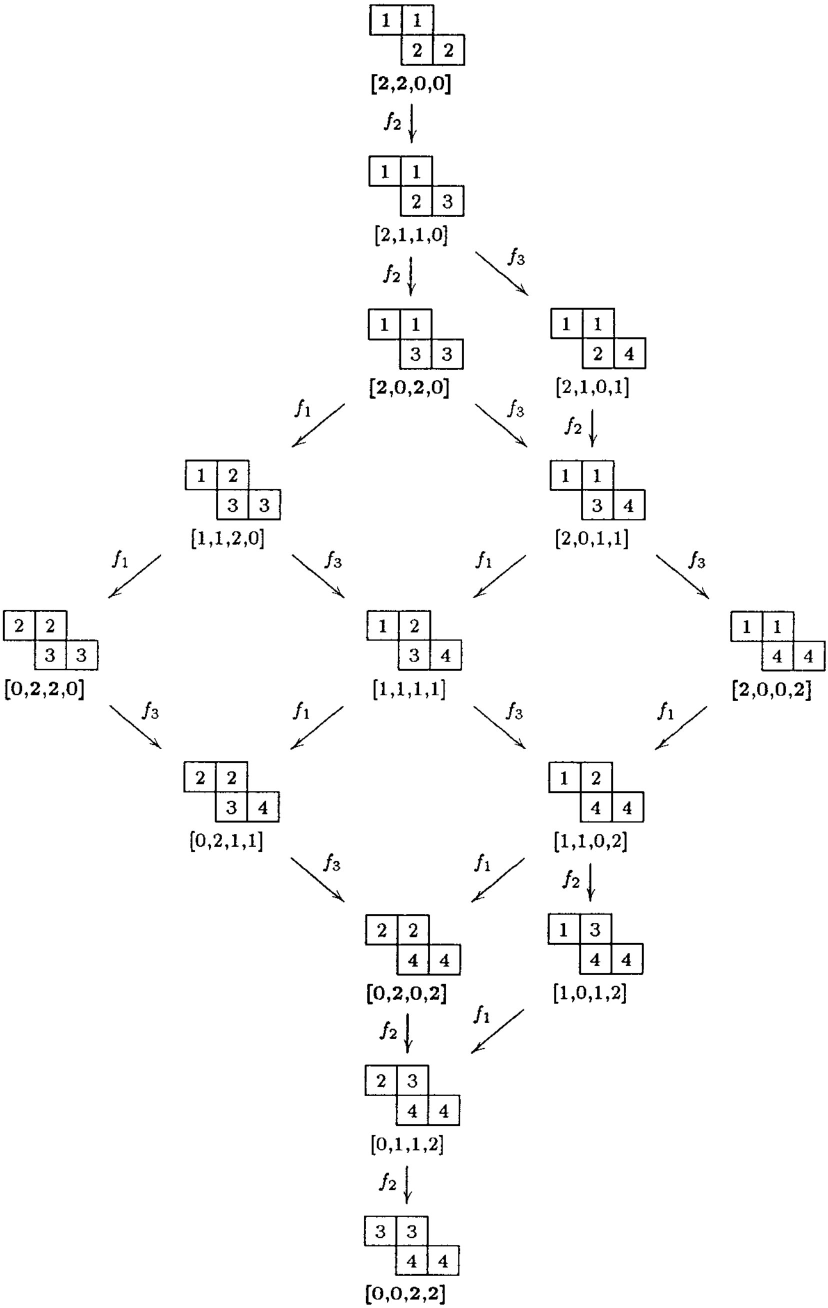


FIG. 1. Quasi-crystal graph of $\mathbf{D}_{22}$ for $\mathscr{U}_0(gl_4)$.

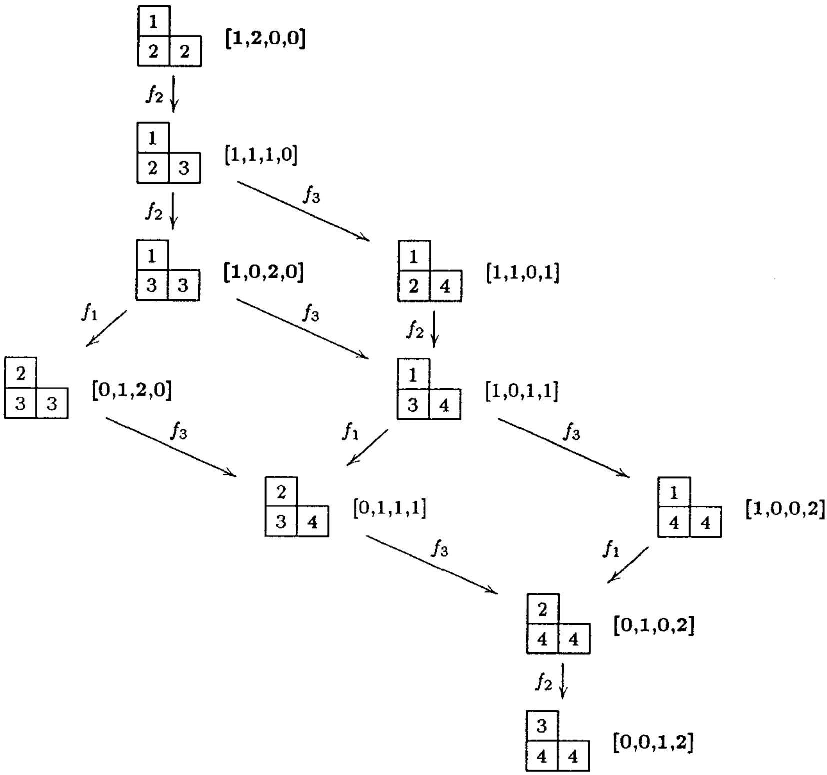


FIG. 2. Quasi-crystal graph of $\mathbf{D}_{12}$ for $\mathscr{U}_0(gl_4)$.

Let K be a composition and $m = [m_1, ..., m_n]$ a monomial which fits in K. The successors of m for this ordering are exactly the monomials of the form

$$f_i(m) = [m_1, ..., m_i - 1, m_{i+1} + 1, ..., m_n] \tag{50}$$

which fits in K. Note that, for each monomial m fitting in K, there is an unique quasi-ribbon word of shape K and of evaluation m (see Section 4). On the quasi-ribbon word, the action of the f_i is to replace the last i by an $i+1$.

This ordering allows to give a simple expression for the partial symmetrizer:

THEOREM 3.29. *Let K be a composition and $m = X^K = [k_1, ..., k_p, 0, ..., 0]$ its associated dominant monomial. Let $\sigma \in \mathfrak{S}_n$. The image of m under the partial symmetrizer $\boldsymbol{\pi}_\sigma$ is given by*

$$\boldsymbol{\pi}_\sigma m = \sum_{\substack{m' \gg \boldsymbol{\sigma} m,\\ m \text{ fits in } K}} m'. \tag{51}$$

Hence, for $K=(2,2)$ with 4 variables, $m=[2,2,0,0]$. Let $\sigma=\sigma_1\sigma_3\sigma_2$, so that $\boldsymbol{\sigma} m=[0,2,0,2]$. Then Fig. 1 shows that

$$\begin{aligned}\boldsymbol{\pi}_\sigma[2,2,0,0]=[2,2,0,0]+[2,1,1,0]+[2,0,2,0]+[2,0,1,1]\\ +[1,1,2,0]+[2,0,0,2]+[1,1,1,1]\\ +[0,2,2,0]+[1,1,0,2]\\ +[0,2,1,1]+[0,2,0,2].\end{aligned}$$

The following proposition will simplify the proof.

PROPOSITION 3.30. *Let K a composition of length p and $m=X^K$ the associated dominant monomial. The image of m under $\boldsymbol{\pi}_\sigma$ depend only on the coset of σ in the quotient $\mathfrak{S}_n/(\mathfrak{S}_p\times\mathfrak{S}_{n-p})$.*

Proof. The monomial m is invariant by all the permutation from the Young subgroup $(\mathfrak{S}_p\times\mathfrak{S}_{n-p})$. But, by Lemma 3.18, if m is invariant under σ_i, it is invariant under $\boldsymbol{\pi}_i$. The proof follows. ∎

Let us prove Theorem 3.29.

Proof. By induction on the length of the permutation σ. First, if $\sigma=\mathrm{Id}$ the formula is true since $m=X^K$ is the greatest monomial fitting in K. Now, suppose that the formula is true for the permutation σ. We have to show the property for any $\sigma_i\sigma$ such that $\ell(\sigma_i\sigma)=1+\ell(\sigma)$ (i is a rise of σ^{-1}).

LEMMA 3.31. *Let $A=\{1,2,...,p\}$ for some $p\leqslant n$ and $\sigma\in\mathfrak{S}_n$. Suppose that i is a rise of σ^{-1}. Then if $i+1$ is in σA one has that i is in σA.*

Proof. Recall that

$$\sigma A=\{\sigma(j)\mid j\in A\}=\{i\mid\sigma^{-1}(i)\in A\}.$$

By hypothesis i is a rise of σ^{-1} and thus $\sigma^{-1}(i)<\sigma^{-1}(i+1)$. But if $i+1\in A$, then $\sigma^{-1}(i+1)\leqslant p$ and then $\sigma^{-1}(i)\leqslant p$. ∎

Let us go Back to the proof of Theorem 3.29. Write

$$\boldsymbol{\sigma} m=\boldsymbol{\sigma} X^k=l=[l_1,...,l_n]. \tag{52}$$

By definition of the quasi-symmetrizing action

$$l=(\sigma\{x_1,...,x^p\})^K. \tag{53}$$

It follows that if the exponent of x_{i+1} in $\boldsymbol{\sigma} X^K$ is non-zero, then the exponent of x_i is also non-zero.

Notice that in the two cases $l_i = l_{i+1} = 0$ on one hand and $l_i \neq 0$ and $l_{i+1} \neq 0$ on the other hand, the transposition $\boldsymbol{\sigma}_i$ leave l invariant. That is,

$$\sigma\{x_1, ..., x_p\} = \sigma_i \sigma\{x_1, ..., x_p\}. \tag{54}$$

By Proposition 3.14 the two permutations σ et $\sigma_i\sigma$ belongs to the same coset modulo $\mathfrak{S}_p \times \mathfrak{S}_{n-p}$. As a consequence of Proposition 3.30, we get that $\boldsymbol{\pi}_\sigma m = \boldsymbol{\pi}_i \boldsymbol{\pi}_\sigma m$ which is the desired formula. It just remains the case $l_i \neq 0$ and $l_{i+1} = 0$.

We suppose that $\boldsymbol{\sigma} m = l = [l_1, ..., l_p]$ with $l_i \neq 0$ and $l_{i+1} = 0$ and that

$$\boldsymbol{\pi}_\sigma m = \sum_{\substack{m' \gg \boldsymbol{\sigma} m, \\ m \text{ fits in } K}} m'. \tag{55}$$

Let us break this sum in a quasi-symmetric part an a remaining part. It is easy to see that the sum of the monomial $m' = [m'_1, ..., m'_n]$ fitting in K and such that

$$m' \gg l \qquad \text{and} \qquad m'_1 + \cdots + m'_{i+1} > l_1 + \cdots + l_i \tag{56}$$

is invariant under $\boldsymbol{\sigma}_i$ and thus invariant under $\boldsymbol{\pi}_i$. It remains the sum of the monomials fitting in K such that

$$m'_1 + \cdots + m'_i = l_1 + \cdots + l_i, \qquad \text{and} \qquad m'_{i+1} = 0. \tag{57}$$

The action of $\boldsymbol{\pi}_i$ on such monomials is to send

$$[m'_1, ..., m'_i, 0, ..., m'_n] \tag{58}$$

to the sum

$$\sum_{u+v=m'_i} [m'_1, ..., u, v, ..., m'_n]. \tag{59}$$

In this way, we get the sum of all monomials q such that

$$\begin{aligned} q_1 + \cdots + q_{i-1} + q_i + q_{i+1} = l_1 + \cdots + l_i \qquad & \text{and} \\ q_1 + \cdots + q_k \geqslant l_1 + \cdots + l_k, \qquad & \text{for} \quad k < i \quad \text{or} \quad k > i+1. \end{aligned} \tag{60}$$

This is precisely the sum of the monomials q fitting in K and such that

$$q \gg \boldsymbol{\sigma}_i l \qquad \text{and} \qquad q_1 + \cdots + q_{i+1} = l_1 + \cdots + l_i \tag{61}$$

This proves the result for the permutation $\boldsymbol{\pi}_i \boldsymbol{\sigma} X^K$, so that the proof follows by induction. ▮

3.5. *Characteristic*

As in the case of the symmetric group, this representation of the Hecke algebra is not faithful if $n > 2$, since the equality

$$T_1 T_2 T_1 = T_2 T_1 T_2 = 0 \tag{62}$$

does not holds in the degenerate Hecke algebra.

In [8], Krob and Thibon showed that there is an analogue of the Frobenius characteristic for the degenerate Hecke Algebra. The characteristic of a module is no longer a symmetric function but a quasi-symmetric one (see [21] for the details). The Frobenius characteristic of this representation is given by the formula

$$\mathrm{ch}_t(\mathbb{C}[x_1, ..., x_n]) = \sum_{m=0}^{n} \frac{t^m}{(1-t)^m} h_{(m,\, n-m)}. \tag{63}$$

It does not give the decomposition of the representation into irreducibles, but rather the composition factor of the module, which is not semi-simple. Since this is a specialization $q=0$ of an action of the generic Hecke algebra, the characteristic is the same as the one of the symmetric group (see [20]).

Moreover, it is possible to give an explicit composition sequence for this representation: let d be an integer. Let $V^d = \mathbb{C}^d[X]$ the homogeneous component of degree d of $\mathbb{C}[X]$. If $m = A^I \in V$ is a monomial, let us call the length of m the cardinal of the support of m. Denote it by $\ell(m)$. Let $V^d_{=l}$ (resp. $V^d_{\leqslant l}$) the subspace of V^d generated by the nomomials of length equal to l (resp. smaller than l). By definition of $\bar{\boldsymbol{\pi}}_i$ (Eq. (37)) $V^d_{\leqslant l}$ is clearly stable under $H_n(0)$. Thus $V^d_{\leqslant l+1}$ is a submodule of $V^d_{\leqslant l}$.

PROPOSITION 3.32. *Let d and l be two integers. Choose a total order $\geqslant$ on the monomial of $V^d_{=l}$ extending the partial order $\gg$,*

$$m^1 \geqslant m^2 \geqslant \cdots \geqslant m^q. \tag{64}$$

Let w^i be the submodule of $V^d_{\leqslant l}$ defined by

$$W^i = \left(V^d_{\leqslant l+1} \oplus \bigoplus_{j \geqslant i} \mathbb{C} m^j \right) \Big/ V^d_{\leqslant l+1}. \tag{65}$$

Then the sequence $(W^i)_i$ is a composition sequence for $V^d_{\leqslant l} / v^d_{\leqslant l+1}$.

Proof. Let us look at the quotient $V^d_{\leqslant l}/V^d_{\leqslant l+1}$. Let $m=[m_1, ..., m_n] \in V^d_{=l}$. The only case where $\pi_i m$ is different of m and not in $\in V^d_{\leqslant l+1}$ is $m_i \neq 0$ and $m_{i+1}=0$. In this case

$$\pi_i[..., m_i, 0, ...] = [..., m_i, 0, ...] + [..., 0, m_i, ...] \qquad \bmod V^d_{\leqslant l+1}. \tag{66}$$

Notice that $[..., m_i, 0, ...] \gg [..., 0, m_i, ...]$. This show that W^i is a submodule of $V^d_{\leqslant l}/V^d_{\leqslant l+1}$. Moreover W^i/W^{i+1} is one dimensional and thus simple. ∎

THEOREM 3.33. *The image of* $H_n(0)$ *in* End$(\mathbb{C}[X])$ *is the quotient of* $H_n(0)$ *by the two-sided ideal* $\mathbb{C}\langle T_\sigma \rangle$ *where* σ *runs over the set of permutations with the pattern* 321.

The proof of this theorem and of the following corollary is postponed to the case of the generic Hecke algebra (see Theorem 5.10).

COROLLARY 3.34. *The set of permutations* σ *such that* $\bar{\pi}_\sigma \neq 0$ *is the set of permutations avoiding the pattern* 321.

4. THE DEGENERATE QUANTUM ENVELOPING ALGEBRA

The aim of this section is to give a representation theoretical interpretation of the two main results of the preceding section. It is provided by the degenerate quantum group $\mathcal{U}_0(gl_N)$ studied by Krob and Thibon in [20–22]. This bialgebra is a specialization of a non-standard two parameter analogue of the universal enveloping algebra of gl_N, which was originally defined by Takeuchi in [47]. The main result of Krob and Thibon is that quasi-symmetric functions are in some sense characters for $\mathcal{U}_0(gl_N)$, in particular the irreducible characters are the quasi-ribbon functions F_I. The reader is referred to these papers for formal definitions.

Let us recall some properties of this algebra. Like the enveloping algebra $U(gl_N)$, the algebra $\mathcal{U}_0(gl_N)$ is generated by three kinds of elements called Chevalley generators:

- raising generators: $(e_i)_{1 \leqslant i \leqslant N-1}$,
- lowering generators: $(f_i)_{1 \leqslant i \leqslant N-1}$,
- diagonal generators: $(k_i)_{1 \leqslant i \leqslant N}$.

The subalgebra $\mathcal{U}_0(\mathfrak{h})$ generated by the k_i is the Cartan subalgebra. We denote by $\mathcal{U}_0(\mathfrak{b}_+)$ the so-called Borel subalgebra generated by e_i and k_i.

Let $(\xi_i)_{1 \leqslant i \leqslant N}$ be the canonical basis of $V = \mathbb{C}^N$. As described in [22], there is a natural morphism from $\mathcal{U}_0(gl_N)$ to $\mathrm{End}_{\mathbb{C}}(V)$ which is called the *fundamental representation* (or *vector representation*) of $\mathcal{U}_0(gl_N)$, denoted by (ρ_V, V). The bialgebra structure on $\mathcal{U}_0(gl_N)$ makes it possible to define tensor products of representations. Let $(\rho_{n,N}, V^{\otimes n})$ be the nth tensor product of the fundamental representation. Note that $\mathcal{U}_0(gl_N)$ is not a Hopf algebra (there is no antipode).

Definition 4.1. A representation of $\mathcal{U}_0(gl_N)$ is said to be *polynomial of degree* n if it is isomorphic to some sub-representation of $(\rho_{n,N}, V^{\otimes n})$.

4.1. *Quasi-Crystal Graph of an Irreducible Module, Weyl Character Formula for* $\mathcal{U}_0(gl_N)$

In [22], using the fact that the action of $\mathcal{U}_0(gl_N)$ commutes with Jimbo's action of $H_n(0)$ on the tensor product, and the classification of the irreducible representations of $H_n(0)$ by Carter [3], Krob and Thibon give a complete description of the irreducible polynomial representation of $\mathcal{U}_0(gl_N)$. Their construction makes use analogues of Young's idempotents in $H_n(0)$. The irreducible modules of $\mathcal{U}_0(gl_N)$ are the image in the tensor product of these analogues. On the way they get a canonical basis for such a module. Let us describe the structure of theses modules. First, we need some combinatorial definitions.

Let A be a finite set (the *alphabet*) and let I denote a composition.

Definition 4.2. A *quasi-ribbon tableau* of shape I is obtained by filling the ribbon diagram associated to I by letters of A in such a way that each row is non-decreasing from left to right and each column is strictly increasing from *top to bottom*. A word is said to be a *quasi-ribbon word* of shape I if it can be obtained by reading from *bottom to top* and from left to right the columns of a quasi-tableau of shape I.

The set of all quasi-ribbon words of shape I is denoted by $\mathrm{QR}(I)$.

For example, the word $\mathbf{u} = abdcbdfef$ is a quasi-ribbon of shape $(3, 1, 3, 2)$ since it is the reading of the first following quasi-ribbon tableau. Conversely the word $aacbabdcd$ is not a quasi-ribbon word. Since the diagram obtained by writing its decreasing factors is not a quasi-ribbon tableau.

quasi-ribbon
$$\begin{array}{|c|c|c|c|c|c|} \hline a & b & b & & & \\ \hline & & c & & & \\ \hline & & d & d & e & \\ \hline & & & & f & f \\ \hline \end{array}$$

not quasi-ribbon
$$\begin{array}{|c|c|c|c|c|c|} \hline a & a & a & & & \\ \hline & & b & & & \\ \hline & & c & \mathbf{b} & c & \\ \hline & & & & d & d \\ \hline \end{array}$$

We are now in a position to describe the irreducible polynomial modules of $\mathscr{U}_0(gl_N)$. The modules $\mathbf{D}_K$ of degree n are indexed by compositions $K \models n$. The basis $\xi_{\mathbf{u}}$ of the module $\mathbf{D}_K$ is indexed by quasi-ribbon $\mathbf{u}$ words of shape K, over the alphabet $A = \{1, ..., n\}$.

Let us describe the action of Chevalley generators on the quasi-ribbon basis. Suppose $\mathbf{u} = u_1 \cdots u_n$ is a quasi-ribbon word. The diagonal generator k_i sends the vector $\xi_{\mathbf{u}}$ to 0 if $\mathbf{u}$ contain the letter i, and otherwise keeps $\xi_{\mathbf{u}}$ unchanged. Let $\mathbf{u}^+$ (resp. $\mathbf{u}^-$) be the word obtained from $\mathbf{u}$ by replacing the last i by a $i+1$ (resp. the last $i+1$ by a i). If there is no such letter $\mathbf{u}^+$ is not defined. The raising operator e_i sends $\xi_{\mathbf{u}}$ to $\xi_{\mathbf{u}^-}$ if $\mathbf{u}^-$ exists and is a quasi-ribbon word of shape K, otherwise it sends $\xi_{\mathbf{u}}$ to 0. The lowering operator f_i send $\xi_{\mathbf{u}}$ to $\xi_{\mathbf{u}^+}$ if $\mathbf{u}^+$ exists and is a quasi-ribbon word of shape K, otherwise it sends $\mathbf{u}$ to 0.

EXAMPLE 4.3. In the module $\mathbf{D}_{(1,2)}$ for $\mathscr{U}_0(gl_4)$ the vector 212 is sent to 0 by f_1 because 222 is not a quasi-ribbon word of shape $(1, 2)$. On the other hand f_2 sends it to 213 (see Fig. 2).

This describes completely the irreducible $\mathscr{U}_0(gl_N)$-module $\mathbf{D}_K$. Figure 2 shows the structure of a $\mathscr{U}_0(gl_N)$-module. We call this graph the quasi-crystal graph $\Gamma_N(I)$. For simplification we only show the action of the f_i. The action of e_i reverses that of the f_i.

Remark that although crystal graphs describe in general only the combinatorial skeleton of a generic module, the quasi-crystal graph $\Gamma_N(I)$ encode the full structure of the $\mathscr{U}_0(gl_N)$-module $\mathbf{D}_K$.

THEOREM 4.4 (Krob and Thibon [22]). *The $\mathscr{U}_0(gl_N)$-module $\mathbf{D}_K$ module is irreducible. Its character is the quasi-ribbon function $F_I(x_1, ..., x_n)$. The $(\mathbf{D}_K)$ form a complete family of irreducible polynomial $\mathscr{U}_0(gl_N)$-modules.*

Recall that in the classical case the character of the irreducible gl_N module $\mathbf{D}_\lambda$ is the Schur function s_λ. Though it has been originally defined as the quotient of two alternants, the Schur function is given by the symmetrization formula [39, 5]: $s_\lambda = \pi_\omega X^\lambda$. In our case Theorem 3.24 can be restated as follows:

THEOREM 4.5 (Weyl Character Formula for $\mathscr{U}_0(gl_N)$). *Let K be a composition. The character of the irreducible module $\mathbf{D}_K$ is given by the formula*

$$\mathrm{ch}(\mathbf{D}_K) = \boldsymbol{\pi}_\omega X^K = F_K. \tag{67}$$

Note 4.6 (Hypoplactic Characters). In fact in [22], Krob and Thibon define characters of $\mathscr{U}_0(gl_N)$ as elements of a quotient of the plactic algebra called the hypoplactic algebra, rather than as quasi-symmetric functions. The hypoplactic algebra admit quasi-ribbon words for basis. They play the same role as tableaux in the plactic algebra. Krob and Thibon show that the subalgebra generated by the hypoplactic quasi-ribbon functions is commutative and isomorphic to the algebra of the quasi-symmetric functions. The reader may look at [43] for a combinatorial study of the hypoplactic algebra.

Proposition 4.8 of the next section allows to lift the action of the Hecke algebra on polynomials in the hypoplactic algebra. That is, we construct hypoplactic divided differences such that the hypoplactic quasi-ribbon function F_K is the image of the unique quasi-ribbon word of shape and evaluation K under the maximal symmetrizer $\boldsymbol{\pi}_\omega$.

We present in our setup an analogue of Demazure's construction [5].

4.2. *Demazure Character Formula for* $\mathcal{U}_0(gl_N)$

The Cartan subalgebra $\mathcal{U}_0(\mathfrak{h})$ acts diagonally on polynomial modules. In the classical case, the eigenspaces for this subalgebra are called weight spaces and the linear forms in $\mathcal{U}_0(\mathfrak{h})^*$ which give the eigenvalues are called weights. In our case, due to the degeneracy, the only possible eigenvalues for k_i are 0 or 1, so that the action of the Cartan subalgebra in not sufficient to define a good notion of weight. We have to use the action of the e_i and f_i and to count multiplicities is some sense. Here we only give a combinatorial definition of the weight of a vector taken out of the previous basis of $\mathbf{D}_K$.

Definition 4.7. Let K be a composition and $\mathbf{u}=u_1\cdots u_n$ a quasi-ribbon word of shape K. The weight of the vector $\xi_\mathbf{u}$ is the evaluation $\mathrm{Eval}(u)$ of the word $\mathbf{u}$, that is, the commutative image of the word $\mathbf{u}$. Otherwise said, if $|\mathbf{u}|_i$ denotes the number of occurrences of the letter i in $\mathbf{u}$, the weight of $\xi_\mathbf{u}$ is the commutative monomial $x_1^{|\mathbf{u}|_1}\cdots x_N^{|\mathbf{u}|_N}$. As usual it will be identified with the pseudo composition $[|\mathbf{u}|_1, ..., |\mathbf{u}|_N]$.

Recall that a monomial $m=A^I$ fits in K if the composition I is finer than K.

Proposition 4.8. *Let K be a composition. Then,* Eval *is a bijective correspondence between quasi-ribbon word of shape K and monomials fitting in K.*

In terms of representation theory, there is a vector of weight m in $\mathbf{D}_K$ if and only if m fits K, and in this case, it is unique, up to constants. Otherwise said, the weight spaces in irreducible representations are one dimensional.

This allows one to transport the combinatorial construction on monomials to vectors. For example, the action of the symmetric group is transported to vectors. This can be seen as the action of the Weyl group $W=\mathfrak{S}_N$.

The ordering of the monomials defined in (3.4) makes sense in representation theory. Indeed, the weight of the image of a vector $\xi_{\mathbf{u}}$ by a raising generator is bigger than the weight of $\xi_{\mathbf{u}}$ itself. It allows us to define highest weights and highest weight vectors: The highest weight of the module $\mathbf{D}_K$ is X^K and the associated vector is $\xi_{\mathbf{u}}$ where $\mathbf{u} = 1^{k_1} 2^{k_2} \cdots N^{k_N}$. By abuse of notation it is denoted by ξ_K. The above properties show that this ordering has a remarkable expression on words: the ordering of vectors of shape K is nothing but the product order on quasi-ribbon words, that is, if $\mathbf{u}$ and $\mathbf{u}'$ are two quasi-ribbons of shape K

$$u_1 \cdots u_N \gg u'_1 \cdots u'_N \qquad \text{iff} \quad u_i > u'_i \quad \text{for all} \quad i \geqslant N. \tag{68}$$

Consequently the quasi-crystal graph is the graph of the product order restricted to the set of quasi-ribbons of shape K.

Gessel shows in [12] that the quasi-ribbon function F_K is the sum of all evaluations of quasi-ribbon words of shape K. It means that the character of an irreducible module is the generating series of the dimensions of its weight spaces, as in the classical cases. In the classical case, this property has a refinement due to Demazure [5]:

Definition 4.9. Let $\mathbf{D}_K$ be an irreducible module of $\mathcal{U}_0(gl_N)$. An *extremal weight* is the image of the highest weight under any element of the Weyl group $W = \mathfrak{S}_N$. To each extremal weight is associated a unique vector up to a constant. These vectors are called *extremal vectors*.

In our case, these are the vectors of weight $m = A^K$ for all $A \in \mathcal{P}_k(X)$ where k is the length of K. They appear in bold-type in the quasi-crystal graphs. Recall that the Borel subalgebra $\mathcal{U}_0(\mathfrak{b}_+)$ is the algebra generated by e_i and k_i.

Definition 4.10. Let ξ be an extremal vector. The $\mathcal{U}_0(\mathfrak{b}_+)$-module generated by ξ is called a Demazure module.

The following theorem is a degenerate analogue of the classical Demazure formula. Since the weight spaces are one dimensional, it gives also a characterisation of bases of Demazure modules analogue to those of [34] (see also [31] for the type A_n).

Theorem 4.11 (Demazure Character Formula for $\mathcal{U}_0(gl_N)$). *Let $\mathbf{D}_K$ an irreducible $\mathcal{U}_0(gl_N)$ module. Suppose that $\sigma \in W$ is a permutation and that $\xi = \boldsymbol{\sigma} \xi_K$ is its associated extremal vector. The generating series of the dimension of the weight spaces, also called character of the Demazure module is given by*

$$\operatorname{ch}(\mathcal{U}_0(\mathfrak{b}_+)\, \xi) = \boldsymbol{\pi}_\sigma X^K. \tag{69}$$

EXAMPLE 4.12. For the algebra $\mathcal{U}_0(gl_4)$ the module $\mathbf{D}_{(1,2)}$ is of dimension 10. Its basis is indexed by the words 212, 213, 313, 214, 323, 314, 324, 414, 424, 434, of respective weights $[1, 2, 0, 0]$, $[1, 1, 1, 0]$, $[1, 0, 2, 0]$, $[1, 1, 0, 1]$, $[0, 1, 2, 0]$, $[1, 0, 1, 1]$, $[0, 1, 1, 1]$, $[1, 0, 0, 2]$, $[0, 1, 0, 2]$, $[0, 0, 1, 2]$.

The extremal weights are $[1, 2, 0, 0]$, $[1, 0, 2, 0]$, $[0, 1, 2, 0]$, $[1, 0, 0, 2]$, $[0, 1, 0, 2]$, $[0, 0, 1, 2]$.

Fix $\sigma = (1423)$. The vector ξ of weight $[1, 0, 0, 2] = \boldsymbol{\sigma}[1, 2, 0, 0]$ generate a Demazure module of dimension 6 whose character is given by $\mathrm{ch}(\mathcal{U}_0(\mathfrak{b}_+)\,\xi) = \boldsymbol{\pi}_\sigma[1, 2, 0, 0] = [1, 2, 0, 0] + [1, 1, 1, 0] + [1, 0, 2, 0] + [1, 1, 0, 1] + [1, 0, 1, 1] + [1, 0, 0, 2]$.

Proof. The work has been almost done in the proof of Theorem 3.29. It remains to identify the weight vectors of the Demazure module. Suppose that $m \gg m'$ are two weights fitting in K and ξ, ξ' are their associated vectors. It is easy to see that ξ is in the image of ξ' under the action of $\mathcal{U}_0(\mathfrak{b}_+)$, so that the Demazure module is the space generated by all vectors of weight greater than $\boldsymbol{\sigma}\xi_K$. The generating series of such weights is well $\boldsymbol{\pi}_\sigma X^K$ according to Theorem 3.29. ∎

5. ACTION OF THE GENERIC HECKE ALGEBRA

5.1. *Main Theorem*

The goal of this section is to construct an action of the generic Hecke algebra generalizing the two preceding actions. Recall that, in the classical case, there are two realisations of the generic Hecke algebra $H_n(q)$ as operator acting on $\mathbb{C}[X]$ (30, 4, 37),

$$T_i \mapsto (q-1)\,\pi_i + \sigma_i, \tag{70}$$

$$T_i \mapsto (1-q)\,\bar{\pi}_i + q\sigma_i. \tag{71}$$

We can also interpolate between $\bar{\boldsymbol{\pi}}$ and $\boldsymbol{\sigma}$ in our case.

THEOREM 5.1. *The operators* $\mathbf{T}_i$ *defined by*

$$\mathbf{T}_i = (1-q)\,\bar{\boldsymbol{\pi}}_i + q\boldsymbol{\sigma}_i = \bar{\boldsymbol{\pi}}_i + q(\boldsymbol{\sigma}_i - \bar{\boldsymbol{\pi}}_i) \tag{72}$$

verify the Hecke relations.

Before proving the theorem let us give the computation rule. To simplify the notation, we only write the rule for $\mathbf{T}_1$, since $\mathbf{T}_i$ acts only on x_i and x_{i+1}.

PROPOSITION 5.2. *Let i, j be two non-zero integers. Then*

$$\mathbf{T}_1[0,0]=q[0,0] \qquad \textit{and} \qquad \mathbf{T}_1[i,j]=q[i,j] \tag{73}$$

$$\mathbf{T}_1[i,0]=(1-q)\sum_{u=1}^{i-1}[i-u,u]+[0,i] \tag{74}$$

$$\mathbf{T}_1[0,i]=(q-1)\sum_{u=1}^{i-1}[i-u,u]+q[i,0]+(q-1)[0,i]. \tag{75}$$

Note 5.3. In the classical case, the divided difference operators commute with multiplication by symmetric polynomials. So it is sufficient to check these identities on a basis of $\mathbb{C}[X]$ as a free module over the ring of symmetric functions. The Grothendieck and Schubert polynomials are helpful in this case [39, 30]. In our case, the quasi-symmetrizing action does not commute with the product, and the ring $\mathbb{C}[X]$ considered as a **Qsym**-module is not free. So the proof is done by a direct calculation, checking the braid relation for all monomials over three variables.

Proof. With the help of the expression (38) and (41) of the operators $\bar{\boldsymbol{\pi}}_i$ and $\boldsymbol{\sigma}_i$, we get

$$\begin{aligned}\mathbf{T}_i^2&=(1-q)^2\,\bar{\boldsymbol{\pi}}_i^2+q(1-q)(\boldsymbol{\sigma}_i\bar{\boldsymbol{\pi}}_i+\bar{\boldsymbol{\pi}}_i\boldsymbol{\sigma}_i)+q^2\\&=-(1-q)^2\,\bar{\boldsymbol{\pi}}_i+q(q-1)\,\boldsymbol{\sigma}_i+q\\&=(q-1)\,\mathbf{T}_i+q,\end{aligned}$$

which proves the quadratic relation. We shall assume the following lemma which can be checked by direct verification.

LEMMA 5.4. *For $i<n-2$ one has*

$$\left.\begin{array}{r}\boldsymbol{\sigma}_i\bar{\boldsymbol{\pi}}_{i+1}\bar{\boldsymbol{\pi}}_i\\+\bar{\boldsymbol{\pi}}_i\boldsymbol{\sigma}_{i+1}\bar{\boldsymbol{\pi}}_i\\+\bar{\boldsymbol{\pi}}_i\bar{\boldsymbol{\pi}}_{i+1}\boldsymbol{\sigma}_i\end{array}\right\}=\left\{\begin{array}{l}\boldsymbol{\sigma}_{i+1}\bar{\boldsymbol{\pi}}_i\bar{\boldsymbol{\pi}}_{i+1}\\+\bar{\boldsymbol{\pi}}_{i+1}\boldsymbol{\sigma}_i\bar{\boldsymbol{\pi}}_{i+1}\\+\bar{\boldsymbol{\pi}}_{i+1}\bar{\boldsymbol{\pi}}_i\boldsymbol{\sigma}_{i+1}\end{array}\right. \tag{76}$$

$$\left.\begin{array}{r}\boldsymbol{\sigma}_i\boldsymbol{\sigma}_{i+1}\bar{\boldsymbol{\pi}}_i\\+\boldsymbol{\sigma}_i\bar{\boldsymbol{\pi}}_{i+1}\boldsymbol{\sigma}_i\\+\bar{\boldsymbol{\pi}}_i\boldsymbol{\sigma}_{i+1}\boldsymbol{\sigma}_i\end{array}\right\}=\left\{\begin{array}{l}\boldsymbol{\sigma}_{i+1}\boldsymbol{\sigma}_i\bar{\boldsymbol{\pi}}_{i+1}\\+\boldsymbol{\sigma}_{i+1}\bar{\boldsymbol{\pi}}_i\boldsymbol{\sigma}_{i+1}\\+\bar{\boldsymbol{\pi}}_{i+1}\boldsymbol{\sigma}_i\boldsymbol{\sigma}_{i+1}.\end{array}\right. \tag{77}$$

Now the braids relation are easy. Indeed, assuming the lemma, the expression

$$\begin{aligned}\mathbf{T}_i\mathbf{T}_{i+1}\mathbf{T}_i = {} & (1-q)^3\,\bar{\boldsymbol{\pi}}_i\bar{\boldsymbol{\pi}}_{i+1}\bar{\boldsymbol{\pi}}_1 \\ & + q(1-q)^2\,(\boldsymbol{\sigma}_i\bar{\boldsymbol{\pi}}_{i+1}\bar{\boldsymbol{\pi}}_i + \bar{\boldsymbol{\pi}}_i\boldsymbol{\sigma}_{i+1}\bar{\boldsymbol{\pi}}_i + \bar{\boldsymbol{\pi}}_i\bar{\boldsymbol{\pi}}_{i+1}\boldsymbol{\sigma}_i) \\ & + q^2(1-q)(\boldsymbol{\sigma}_i\boldsymbol{\sigma}_{i+1}\bar{\boldsymbol{\pi}}_i + \boldsymbol{\sigma}_i\bar{\boldsymbol{\pi}}_{i+1}\boldsymbol{\sigma}_i + \bar{\boldsymbol{\pi}}_i\boldsymbol{\sigma}_{i+1}\boldsymbol{\sigma}_i) \\ & + q^3\boldsymbol{\sigma}_i\boldsymbol{\sigma}_{i+1}\boldsymbol{\sigma}_i\end{aligned}$$

appears to be symmetric in i and $i+1$ and thus equal to $\mathbf{T}_{i+1}\mathbf{T}_i\mathbf{T}_{i+1}$. ∎

5.2. *Yang–Baxter Elements in the Hecke Algebra*

The aim of this section is to recall some classical fact on the Yang–Baxter equation (see, for example, [7]). Let us define the two elements

$$\square_i = T_i + 1 \qquad \text{and} \qquad \nabla_i = T_i - q. \tag{78}$$

They verify the commutation rules

$$\square_i T_i = T_i \square_i = q\,\square_i \qquad \text{and} \qquad \nabla_i T_i = T_i \nabla_i = -\nabla_i \tag{79}$$

together with the relation

$$\square_i^2 = (q+1)\,\square_i \qquad \text{and} \qquad \nabla_i^2 = -(q+1)\,\nabla_i \tag{80}$$

which will be useful to construct q-idempotents. However, they do not satisfy the braid relation but a deformation of it, called the Yang–Baxter relation,

$$\square_i\left(\square_{i+1} - \frac{q}{1+q}\right)\square_i = \square_{i+1}\left(\square_i - \frac{q}{1+q}\right)\square_{i+1}, \tag{81}$$

$$\nabla_i\left(\nabla_{i+1} + \frac{q}{1+q}\right)\nabla_i = \nabla_{i+1}\left(\nabla_i + \frac{q}{1+q}\right)\nabla_{i+1}. \tag{82}$$

This relation makes it possible to define elements $\square_\sigma$ and ∇_σ for any permutation σ: by induction, start with $\square_{\mathrm{Id}} = \nabla_{\mathrm{Id}} = 1$ where Id is the identity of the symmetric group, and if i is a rise of σ'^{-1}, so that $\sigma = \sigma_i\sigma'$ is a reduced expression for σ, let $u = (\sigma')^{-1}(i)$ and $v = (\sigma')^{-1}(i+1)$. Then, thanks to Yang–Baxter relations, the elements

$$\square_\sigma = \left(\square_i - q\,\frac{[v-u-1]_q}{[v-u]_q}\right)\square_{\sigma'} \qquad \text{and}$$

$$\nabla_\sigma = \left(\nabla_i + q\,\frac{[v-u-1]_q}{[v-u]_q}\right)\nabla_{\sigma'}$$

are independent of the choice of σ' and i. Here the q-integers are defined by $[c]_q = (1-q^c)/(1-q)$.

The interest of the so-defined operators $\square_\omega$ and ∇_ω is that there are actually q-analogues of the full symmetrizer and anti-symmetrizer (cf. [7]):

THEOREM 5.5. *Let ω be the maximal permutation of $\mathfrak{S}_n$. Then*

$$\square_\omega = \sum_{\sigma \in \mathfrak{S}_n} T_\sigma \qquad \textit{and} \qquad \nabla_\omega = \sum_{\sigma \in \mathfrak{S}_n} (-q)^{\ell(\omega\sigma)} T_\sigma \tag{83}$$

We will also makes use of the two factorizations

$$\square_\omega = \square_{\omega'} (1 + T_{n-1} + T_{n-1} T_{n-2} + \cdots + T_{n-1} \cdots T_1), \tag{84}$$

$$\square_\omega = (1 + T_{n-1} + T_{n-2} T_{n-1} + \cdots + T_1 T_2 \cdots T_{n-1}) \, \square_{\omega'}, \tag{85}$$

where ω' is the maximal permutation of $\mathfrak{S}_{n-1}$.

5.3. *q-Quasi-Symmetrizing Operators*

This subsection is devoted to the study of the quasi-symmetrizing action of the Yang–Baxter elements.

THEOREM 5.6. *The image of the full q-symmetrizing operator $\boxdot_\omega$, as map on $\mathbb{Z}[q][X]$ is the space of quasi-symmetric functions.*

Moreover, if we take coefficients in $\mathbb{C}(q)$, the operator $(1/[n]_q!) \boxdot_\omega$ is an idempotent whose image is the space of quasi-symmetric functions.

As usual, the q-factorial if given by $[n]_q! = \prod_{i=1}^n [i]_q$.

Proof. Let i be an integer. Let $\sigma' = \sigma_i \omega$, then $\square_\omega = \square_i \square_{\sigma'}$. Since $\boldsymbol{\sigma}_i \boxdot_i = \boxdot_i$, on has $\boldsymbol{\sigma}_i \boxdot_\omega f = \boxdot_\omega f$, for any f and i, and consequently that $\boxdot_\omega f$ is quasi-symmetric.

Conversely, if f is quasi-symmetric, then $\mathbf{T}_i f = qf$. And thus,

$$\boxdot_\omega f = \sum_{\sigma \in \mathfrak{S}_n} \mathbf{T}_\sigma f = \sum_{\sigma \in \mathfrak{S}_n} q^{\ell(\sigma)} f. \tag{86}$$

Since $[n]_q!$ is the generating function of permutations counted by length, the proof is done. ∎

As a consequence of the proof we get the following characterization of quasi-symmetric polynomials:

PROPOSITION 5.7. *A polynomial f if quasi-symmetric if and only if $\mathbf{T}_i f = qf$ for all $i \in \{1, ..., n\}$.*

On the other hand, the q-anti-symmetrizing operator $\boldsymbol{\nabla}_\omega$ is more drastic:

THEOREM 5.8. *Suppose that* $n > 2$. *Then* $\boldsymbol{\nabla}_\omega = 0$.

Proof. First, the operator $\boldsymbol{\nabla}_\omega$ factorizes as

$$\boldsymbol{\nabla}_1 \left(\boldsymbol{\nabla}_2 + \frac{q}{1+q} \right) \boldsymbol{\nabla}_1 \left(\boldsymbol{\nabla}_3 + \frac{1+q}{1+q+q^2} \right) \left(\boldsymbol{\nabla}_2 + \frac{q}{1+q} \right) \cdots, \tag{87}$$

thus it is sufficient to prove the theorem for $n = 3$. Note that

$$\boxdot_i \boldsymbol{\nabla}_i = \boldsymbol{\nabla}_i \boxdot_i = 0. \tag{88}$$

Expanding $\boxdot_i$ we get

$$(\bar{\boldsymbol{\pi}}_i + q(1 + \boldsymbol{\sigma}_i + \bar{\boldsymbol{\pi}}_i)) \, \boldsymbol{\nabla}_i = 0. \tag{89}$$

Hence we have proven the following lemma:

LEMMA 5.9. *Let* $i \in \{1, ..., n-1\}$. *Then* $\boldsymbol{\sigma}_i \boldsymbol{\nabla}_i = -\boldsymbol{\nabla}_i$.

To continue, remark that a reduced expression for $\boldsymbol{\nabla}_\omega$ can start with either $\boldsymbol{\nabla}_1$ or $\boldsymbol{\nabla}_2$. Thus $\boldsymbol{\sigma}_1 \boldsymbol{\nabla}_\omega = \boldsymbol{\sigma}_2 \boldsymbol{\nabla}_\omega = -\boldsymbol{\nabla}_\omega$. But also $\boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_1 \boldsymbol{\nabla}_\omega = -\boldsymbol{\nabla}_\omega$. Now suppose that A is a subset of $\{1, 2, 3\}$. Then, either two of the integers 1, 2, 3 are in A or else two of them are in the complementary of A. Thus the transposition which exchanges these two integers fixes A. As a consequence each monomial is fixed at least by the quasi-symmetrizing action of one of the three transpositions $\sigma_1 = (1, 2)$, $\sigma_2 = (2, 3)$ or $\sigma_1 \sigma_2 \sigma_1 = (1, 3)$. We conclude that for all $f \in \mathbb{C}[X]$, we have $\boldsymbol{\nabla}_\omega f = 0$. ∎

Since this relation does not hold in the Hecke algebra, once more, this is not a faithful representation. It is possible to give a complete characterization of the kernel:

THEOREM 5.10. *The image of* $H_n(q)$ *in* $\mathrm{End}(\mathbb{C}[X])$ *is the quotient of* $H_n(q)$ *by the ideal generated by* $(\nabla_{(i,\, i+2)})$ *where* $i = 1 \cdots n-2$.

The family $(\boldsymbol{\nabla}_\sigma)$ *where* σ *runs over the set of all permutations avoiding the pattern* 321 *is a basis of the image of* $H_n(q)$ *in* $\mathrm{End}(\mathbb{C}[X])$.

Proof. The equality $\boldsymbol{\nabla}_{(i,\, i+2)} = 0$ is already proved in Theorem 5.8. Hence the family $(\boldsymbol{\nabla}_\sigma)$ with σ avoiding the pattern 321 generate the image.

Let E be the space of polynomial in X such that no monomials contain any squared variables. With our notations there are monomials of the form A^K with $A \in \mathscr{P}(X)$ and $K = (1, 1, ..., 1)$. As a Hecke algebra module, it is isomorphic to the space generated by subsets of X or to the space $V^{\otimes n}$ where $V = \mathbb{C}\xi_0 + \mathbb{C}\xi_1$ is a two dimensional space. It is known that the dimension of the image of $H_n(q)$ in $\mathrm{End}(V^{\otimes n})$ is the number of permutations avoiding the pattern 321 [3]. Thus the announced family is a basis.

Finally, the first part of the theorem follows immediately, since the equality $\nabla_{(i, i+2)} = 0$ allows us to express all the actions on the basis. ∎

Note that the number of permutation avoiding 321 in $\mathfrak{S}_n$ is the Catalan number $C_n = \frac{1}{n-1}\binom{2n}{n}$. For example, in 4 variables, the image of $H_n(q)$ in End($\mathbb{C}[X]$) is of dimension 14. A basis for this image is (∇_σ) for $\sigma \in \{1234$, 1243, 1324, 1342, 1423, 2134, 2143, 2314, 2341, 2413, 3124, 3142, 3412, $4123\}$. Remark that ∇_{4231} is not in the kernel.

As an equivalent property we can give the character of this action of the Hecke algebra:

Proposition 5.11. *The Frobenius characteristic of the quasi-symmetrizing action of the Hecke algebra is*

$$\mathrm{ch}_t(\mathbb{C}[x_1, ..., x_n]) = \sum_{m=0}^{n} \frac{t^m}{(1-t)^m} h_{(m, n-m)}. \tag{90}$$

Proof. This is a well known fact that the characteristic of an action of the generic Hecke algebra is the characteristic of the action of the symmetric group obtained by letting $q = 1$. Indeed, the generic Hecke algebra is isomorphic to the algebra of the symmetric group, so that the representation theory are the same. Consequently, the characteristic of the irreducible q-Specht module $V(\lambda)$ associated with the partition λ remains to be the Schur function s_λ [7].

But if E is a module, its decomposition into irreducibles is obtained by decomposing its character as

$$\chi_E(q) = \sum_\lambda c_\lambda \chi^\lambda(q). \tag{91}$$

The integers c_λ do not depend on q. So they can be computed by specifying $q = 1$. Thus the characteristic of the representation of generic Hecke algebra is equal to the characteristic of the action of the symmetric group which is given by Proposition 3.9. ∎

Note that, as shown in [8], this formula is still valid for $q = 0$. It does not give the decomposition of the representation in irreducible, but rather the factor composition of the module.

6. HALL–LITTLEWOOD FUNCTIONS

6.1. *Outline of the Classical Theory*

We first recall some facts of the classical theory. Our notations will be essentially those of [38], to which the reader is referred for more details.

Let $\Delta_n(q)$ denote the q-alternant $\prod_{i<j\leqslant n}(qx_i-x_j)$. Then, on the one hand, the Hall–Littlewood polynomial $Q_\lambda(x_1, ..., x_n; q)$ indexed by a partition λ of length $\leqslant n$ is defined by [35]

$$Q_\lambda = \frac{(1-q)^{\ell(\lambda)}}{[m_0]_q!} \sum_{\sigma\in\mathfrak{S}_n} \sigma_\sigma\left(x^\lambda \frac{\Delta_n(q)}{\Delta_n(1)}\right), \tag{92}$$

where $m_0 = n-\ell(\lambda)$ and σ_σ is the usual action of the permutation σ on polynomials.

On the other hand, it is shown in [7] that if $\square_\omega$ denote the classical q-symmetrizing operator $\square_\omega = \sum_{\sigma\in\mathfrak{S}_n} T_\sigma$, then

$$Q_\lambda(x_1, ..., x_n; q^{-1}) = q^{-\binom{N}{2}} \frac{(1-q^{-1})^{\ell(\lambda)}}{[m_0]_{q^{-1}}!} \square_\omega(x^\lambda). \tag{93}$$

The normalization factor $1/[m_0]_q!$ is to ensures stability with respect to the adjunction of variables, and if we denote by X the infinite set $X=\{x_1, x_2, ..., \}$ then $Q_\lambda(X; q) = \lim_{n\to\infty} Q_\lambda(x_1, ..., x_n; q)$.

The P-functions are defined by

$$P_\lambda(X; q) = \frac{1}{(1-q)^{\ell(\lambda)}[m_1]_q!\cdots[m_n]_q!} Q_\lambda(X; q), \tag{94}$$

where m_i is the multiplicity of the part i in λ.

We have the specializations: $P_\lambda(X; 0)$ is equal to the Schur function s_λ and $P_\lambda(X; 1)$ is equal to the monomial function m_λ,

We consider these functions as elements of the algebra $\mathrm{Sym} = \mathrm{Sym}(X)$ of symmetric functions with coefficients in $\mathbb{C}(q)$. There is a scalar product $\langle\ ,\ \rangle$ on Sym, for which the Schur functions s_λ form an orthonormal basis. We denote by $(Q'_\mu(X; q))$ the adjoint basis of $P_\lambda(X; q)$ for this scalar product. It is easy to see that $Q'_\mu(X; q)$ is the image of $Q_\mu(X; q)$ by the ring homomorphism $p_k \mapsto (1-q^k)^{-1} p_k$ (in λ-ring notation, $Q'_\mu(X; q) = Q(X/(1-q); q)$). In the Schur basis,

$$Q'_\mu(X; q) = \sum_\lambda K_{\lambda\mu}(q)\, s_\lambda(X), \tag{95}$$

where the $K_{\lambda\mu}(q)$ are the so-called Kotska–Foulkes polynomials. Lascoux and Schützenberger showed that the polynomial $K_{\lambda\mu}(q)$ is the generating function of a statistic c called *charge* on the set $\mathrm{Tab}(\lambda, \mu)$ of Young tableaux of shape λ and weight μ [38].

6.2. *Quasi-Symmetric Hall–Littlewood Functions*

By analogy, we introduce the following quasi-symmetric analogues of the P-functions:

DEFINITION 6.1. Let $K=(k_1, ..., k_p)$ be a composition and $X^K=x_1^{k_1}\cdots x_p^{k_p}$. The *quasi-symmetric Hall–Littlewood function* G_K is defined by

$$G_K(x_1, ..., x_n; q)=\frac{1}{[p]_q!\,[n-p]_q!}\,\boxdot_\omega(X^K). \tag{96}$$

We have the following specializations: $G_K(X; 0)$ is the quasi-ribbon function F_K and $G_K(X; 1)$ is the quasi-monomial function M_K.

For instance $G_{(2,1)}(x_1, x_2, x_3; q)=x_1^2\,x_2+x_1^2\,x_3+x_2^2\,x_3+(1-q)\,x_1x_2x_3$. On this example, we verify that at $q=0$, one has $G_{(2,1)}(x_1, x_2, x_3; 0)=F_{(2,1)}$ and at $q=1$, one has $G_{(2,1)}(x_1, x_2, x_3; 1)=M_{(2,1)}$.

PROPOSITION 6.2. *Let K be a composition of length p. Then*

$$G_K(x_1, ..., x_n; q)=\sum_{\sigma\in\mathfrak{S}_{n/p}}\mathbf{T}_\sigma X^K. \tag{97}$$

Proof. Recall that $\mathfrak{S}_{n/p}$ is exactly the set of permutations σ such that

$$\sigma(1)<\sigma(2)<\cdots<\sigma(p)\qquad\text{and}\qquad\sigma(p+1)<\cdots<\sigma(n). \tag{98}$$

But a permutation σ factorizes in a unique way as $\sigma=\mu\mu'$ with $\mu\in\mathfrak{S}_{n/p}$, $\mu'\in\mathfrak{S}_p\times\mathfrak{S}_{n-p}$. Consequently we have the following factorization of the full q-symmetrizer,

$$\boxdot_\omega=\Bigg(\sum_{\sigma\in\mathfrak{S}_{n/p}}\mathbf{T}_\sigma\Bigg)\Bigg(\sum_{\sigma\in\mathfrak{S}_p\times\mathfrak{S}_{n-p}}\mathbf{T}_\sigma\Bigg). \tag{99}$$

Now, since K is of length p, for each $\sigma\in\mathfrak{S}_p\times\mathfrak{S}_{n-p}$ we have $\mathbf{T}_\sigma X^K=q^{\ell(\sigma)}X^K$. This ends the proof since $[n]_q!$ is the generating series of the permutations of $\mathfrak{S}_n$ enumerated by their length. ∎

COROLLARY 6.3. *The polynomial $G_K(x_1, ..., x_n; q)$ is quasi-symmetric with coefficients in $\mathbb{Z}[q]$. More precisely, the expansion of the quasi-symmetric Hall–Littlewood polynomials on the monomial basis is of the form $G_K=M_K+\sum_{J\succcurlyeq K}a_JM_J$ where a_J is a polynomial in q with integer coefficients.*

COROLLARY 6.4. *The family $(G_K)_{\ell(K)\leqslant n}$ is a basis of the space of quasi-symmetric polynomials with coefficients in $\mathbb{C}(q)$.*

We will prove in the next section that in fact (G_K) remains a basis even if we take coefficients in $\mathbb{Z}[q]$. The transition matrix is upper unitriangular (i.e., triangular with 1 on the diagonal). We will give an explicit formula for the polynomial a_J in the proof of Theorem 6.6.

The coefficient $1/[p]_q!\,[n-p]_q!$ is here to ensure that G_K has the stability property:

PROPOSITION 6.5. *Let K be of length $p \leqslant n$. Then,*

$$G_K(x_1, ..., x_n, 0; q) = G_K(x_1, ..., x_n; q). \tag{100}$$

Proof. Let us denote $\boxdot_\omega^{(i)}$ the q-symmetrizing operator associated with the maximal permutation of $\mathfrak{S}_i$. The proposition is equivalent to the expression

$$(\boxdot_\omega^{(n+1)} X^K)_{/x_{n+1}=0} = [n-p+1]_q\,(\boxdot_\omega^{(n)} X^K). \tag{101}$$

Since we need this in the sequel of the proof, let us start by showing the property in the particular case $n = p$.

Let $m = X^K = [k_1, ..., k_p, 0]$. Then $\boxdot_\omega^{(p)} m = [p]_q!\, m$. Using the factorization (84) of $\boxdot_\omega^{(p+1)}$, we have

$$\boxdot_\omega^{(p+1)} m = [p]_q!\,(1 + \mathbf{T}_p + \mathbf{T}_{p-1}\mathbf{T}_p + \cdots + \mathbf{T}_1\mathbf{T}_2\cdots\mathbf{T}_p)\, m.$$

But,

$$\mathbf{T}_p m = (1-q) \sum_{v=1}^{k_n} [k_1, ..., k_n - v, v] + q[k_1, ..., 0, k_p].$$

Thus for $r \leqslant p$, we get $(\mathbf{T}_r \cdots \mathbf{T}_{p-1}\mathbf{T}_p m)_{/x_{n+1}=0} = 0$. Finally, we have proved $(\boxdot_\omega^{(p)} m)_{/x_{p+1}=0} = \boxdot_\omega^{(p)} m$, which is the result for $n = p$.

Let us prove it for any integer $n > p$. Let $f' = \boxdot_\omega^{(n)} m$ together with $f = (\boxdot_\omega^{(n+1)} m)_{/x_{n+1}=0}$. The factorization of $\boxdot_\omega^{(n+1)}$ gives

$$f = ((1 + \mathbf{T}_n + \mathbf{T}_{n-1}\mathbf{T}_n + \mathbf{T}_1\mathbf{T}_2\cdots\mathbf{T}_n) f')_{/x_{n+1}=0}. \tag{102}$$

Now suppose that g is a monomial of the form $[g_1, ..., g_n, 0]$ with $g_n \neq 0$. Then, all the monomials appearing in $\bar{\boldsymbol{\pi}}_n g$ have a non-zero final part, and consequently $(\bar{\boldsymbol{\pi}}_n g)_{x_{n+1}=0} = 0$. It follows

$$(\mathbf{T}_n f)_{/x_{n+1}=0} = q\boldsymbol{\sigma}_n(f'_{/x_n=0}) = qf'_{/x_n=0}$$

so that Eq. (102) reads

$$f = f' + q(1 + \mathbf{T}_{n-1} + \mathbf{T}_{n-1}\mathbf{T}_{n-2} + \mathbf{T}_1\mathbf{T}_2\cdots\mathbf{T}_{n-1})(f'_{/x_n=0}).$$

Now, if we suppose that the result is true for $n-1$ we get that

$$f'_{/x_n=0} = [n-p]_q\, \boxdot_\omega^{(n-2)} X^K.$$

Putting the last two equations together we have

$$f=f'+q[n-p]_q\,\boxdot_{\omega}^{(n-1)}X^K=(1+q[n-p]_q)f'.$$

Since $[n-p+1]_q=1+q[n-p]_q$ we get the property for $n+1$, and the proof follows by induction. ∎

This makes it possible to take the limits when $n\to+\infty$. We call them quasi-symmetric Hall–Littlewood functions. Due to the expansion formula, the family of quasi-symmetric Hall–Littlewood functions is a basis of the algebra quasi-symmetric functions.

Moreover, it is possible to give an explicit expansion of the Hall–Littlewood functions in the classical bases. The most striking expansion is in the quasi-ribbon basis.

THEOREM 6.6. *The expansion of G_I in the quasi-ribbon basis is given by*

$$G_I=\sum_{J\geqslant I}(-1)^{\ell(J)-\ell(I)}\,q^{s(I,J)}F_J, \tag{103}$$

where $s(I,J)$ is defined as follows. Let $(k_1,...,k_p)$ be the refining composition Bre(J,I). *Then $s(I,J)=(k_1-1)+2(k_2-1)+\cdots+p(k_p-1)$.*

EXAMPLE 6.7. Let us compute $G_{(1,1,2,1)}$ on $X=\{x_1,x_2,x_3,x_4,x_5\}$. Start with $m=x_1x_2x_3^2x_4=[1,1,2,1,0]$. Since m is symmetric for $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, and $\boldsymbol{\sigma}_3$, the operators T_1, T_2, and T_3 multiply m by q. Therefore $\boxdot_{\omega'}m=[4]_q!\,m$,

$$\begin{aligned}\mathbf{T}_4(m)&=[1,1,2,0,1]\\ \mathbf{T}_3\mathbf{T}_4(m)&=[1,1,0,2,1]+(1-q)[1,1,1,1,1]\\ \mathbf{T}_2\mathbf{T}_3\mathbf{T}_4(m)&=[1,0,2,1,1]+q(1-q)[1,1,1,1,1]\\ \mathbf{T}_1\mathbf{T}_2\mathbf{T}_3\mathbf{T}_4(m)&=[0,1,1,2,1]+q^2(1-q)[1,1,1,1,1]\end{aligned}$$

From the factorization

$$\boxdot_{\omega}=(1+\mathbf{T}_4+\mathbf{T}_3\mathbf{T}_4+\mathbf{T}_2\mathbf{T}_3\mathbf{T}_4+\mathbf{T}_1\mathbf{T}_2\mathbf{T}_3\mathbf{T}_4)\,\boxdot_{\omega'},$$

we get

$$\begin{aligned}\boxdot_{\omega}([1,1,2,1,0])=[4]_q!([1,1,2,1,0]&+[1,1,2,0,1]\\ &+[1,1,0,2,1]+[1,0,1,2,1]+[0,1,1,2,1]\\ &+(1-q)(1+q+q^2)[1,1,1,1,1]).\end{aligned}$$

Or, using the quasi-monomial basis of **Qsym**

$$\begin{aligned}\boxdot_\omega([1,1,2,1,0]) &= [4]_q!\,(M_{(1,1,2,1)}+(1-q^3)\,M_{(1,1,1,1,1)})\\ &= [4]_q!\,(F_{(1,1,2,1)}-q^3F_{(1,1,1,1,1)}),\end{aligned}$$

so that $G_{(1,1,2,1)}=F_{(1,1,2,1)}-q^3F_{(1,1,1,1,1)}$.

Similarly, one would find

$$\begin{aligned}G_{(3,2)} = F_{(3,2)} &- q\,F_{(2,1,2)} - qF_{(1,2,2)} + q^2\,F_{(1,1,1,2)} - q^2\,F_{(3,1,1)}\\ &+ q^3F_{(2,1,1,1)} + q^3F_{(1,2,1,1)} - q^4F_{(1,1,1,1,1)}.\end{aligned}$$

LEMMA 6.8. *Let $I=(i_1, ..., i_r)\succcurlyeq J=(j_1, ..., j_p)$ be two compositions. Let $K=\mathrm{Bre}(I, J)\models r$. The coefficient of $F_{(1^r)}$ in G_K is the same as the one of F_I in G_J. That is, the coefficient of F_I in G_J depends only on* $\mathrm{Bre}(I, J)$.

Proof. Let ϕ_I denote the map which takes a pseudo composition $m=[m_1, ..., m_n]$ of r and sends it to $\phi_I(m)=[m'_1, ..., m'_n]$ defined by

$$m'_i=\begin{cases} i_{m_1+\cdots+m_{i-1}+1}+\cdots+i_{m_1+\cdots+m_i} & \text{if}\quad m_i\neq 0,\\ 0, & \text{otherwise.}\end{cases}$$

For example, with $I=(1, 2, 5, 4, 1, 2)$, we have

$$\phi_I[1,0,0,2,0,1,2]=[1,0,0,7,0,4,3] \qquad \text{and}$$
$$\phi_I[3,0,0,1,2]=[8,0,0,4,3]$$

Remark that $\phi_I([1^r, 0, ..., 0])=X^I$, and that I is finer than any exponent composition appearing in $\phi_I(m)$. For a given m, we have that

$$\mathbf{T}_u\phi_I(m)=\phi_I(\mathbf{T}_u m)+\text{some monomials not smaller than } I.$$

Moreover, if l is a monomial not smaller than I, then $\mathbf{T}_u l$ contains no monomials smaller that I. So, by induction,

$$\mathbf{T}_\sigma\phi_I(m)=\phi_I(\mathbf{T}_\sigma m)+\text{some monomials not smaller than } I,$$

for any permutation σ. Finally, we find that

$$\boxdot_\omega\phi_I(m)=\phi_I(\boxdot_\omega m)+\text{monomials not smaller than } I.$$

Back from monomials to quasi-ribbon function, the proof is done, since F_I is the sum of all monomials smaller than I. ∎

EXAMPLE 6.9. Let $m=[3, 0]$. We have

$$\mathbf{T}_1 m=[0, 3]+(1-q)[1, 2]+(1-q)[2, 1].$$

Let now $I=(3, 1, 2)$, so that $\phi_I(m)=[6, 0]$. Thus

$$\mathbf{T}_1 \phi_I(m)=[0, 6]+(1-q)\left(\sum_{u=1}^{5} [u, 6-u]\right).$$

We verify that the only monomials of the form $[u, 6-u]$ smaller than I are $[3, 3]=\phi_I([1, 2])$ and $[4, 2]=\phi_I([2, 1])$.

Proof of Theorem 6.6. Let $K=(k_1, ..., k_p)\models r$. By virtue of the preceding lemma, it is sufficient to prove that the coefficient of $[1^r]$ in G_K is equal to

$$\begin{aligned}(1-q)^{k_1-1}\,(1-q^2)^{k_2-1}\,(1-q^3)^{k_3-1}\cdots(1-q^p)^{k_p-1}\\ =(1-q)^{r-p}\,1^{k_1-1}(1+q)^{k_2-1}\,(1+q+q^2)^{k_3-1}\cdots\\ \times(1+q+\cdots+q^{p-1})^{k_p-1}\end{aligned}$$

We will prove this in two steps.

LEMMA 6.10. *The coefficient of the monomial* $[1^{r+1}]$ *in the polynomial* $G_{(K, 1)}$ *is equal to the one of* $[1^r]$ *in* G_K.

LEMMA 6.11. *The coefficient of* $[1^{r+1}]$ *in* $G_{(k_1, ..., k_p+1)}$ *is equal to the coefficient of* $[1^r]$ *in* G_K, *multiplied by* $(1-q^p)=(1-q)(1+q+\cdots q^{p-1})$.

These two lemmas will prove the theorem by induction. ∎

Let us now prove the lemmas.

Proof of Lemma 6.10. By Proposition 6.2, we have

$$G_{(K, 1)}=\sum_{\sigma\in\mathfrak{S}_{r+1/p+1}} \mathbf{T}_\sigma X^{(K, 1)}.$$

Recall that for $\sigma\in\mathfrak{S}_{r+1/p+1}$, the operator $\mathbf{T}_\sigma$ has a reduced word of the form

$$(\mathbf{T}_{\sigma(1)-1}\cdots\mathbf{T}_2\mathbf{T}_1)(\mathbf{T}_{\sigma(2)-1}\cdots\mathbf{T}_3\mathbf{T}_2)\cdots(\mathbf{T}_{\sigma(p+1)-1}\cdots\mathbf{T}_{p+2}\mathbf{T}_{p+1})$$

with $\sigma(1)<\sigma(2)<\cdots<\sigma(p)<\sigma(p+1)$. But the $(r+1)$st variable will have a zero exponent under $\mathbf{T}_\sigma X^K$ unless $\sigma(p+1)=r+1$. Now

$$\mathbf{T}_r\cdots\mathbf{T}_{p+2}\mathbf{T}_{p+1}X^K=[k_1\cdots k_p, 0, ..., 0, 1].$$

Keeping the last variable aside, it just remains to compute the image of X^K under $\sum_{\sigma \in \mathfrak{S}_{r/p}} \mathbf{T}_\sigma$, which is G_K as well. The lemma is proved. ∎

Proof of Lemma 6.11. Let us denote $K \triangleright 1$ the composition $(k_1, ..., k_p+1)$ and $\hat{K} = (k_1, ..., k_{p-1})$. Let ω denote the maximal permutation of $\mathfrak{S}_{r+1}$ and ω' the maximal permutation of $\mathfrak{S}_r$. We want to compute the coefficient of $[1^{r+1}]$ in $\boxdot_\omega[\hat{K}, k_p+1, 0, ..., 0]$.

Using (84), we have just to compute the image of $X^{K \triangleright 1}$ under $\mathbf{T}_r \cdots \mathbf{T}_s$. There are two cases. If $s > p$, since all the exponents of $x^{p+1}, ..., x^{r+1}$ are zero in $X^{K \triangleright 1}$ the action of the successive $\mathbf{T}_i$ is only to multiply it by q. Now if $s \leqslant p$, the action of the $\mathbf{T}_i$ for $i < p$ is again to multiply by q. Now since

$$\mathbf{T}_p[\hat{K}, k_p+1, 0] = [\hat{K}, 0, k_p] + 1 + (1-q) \sum_{u+v=k_p+1,\ u \neq 0,\ v \neq 0} [\hat{K}, u, v]$$

we get the result

$$\mathbf{T}_r \cdots \mathbf{T}_s[\hat{K}, k_p+1, 0, ..., 0] = \begin{cases} q^{p-s} \displaystyle\sum_{\substack{u_p + \cdots + u_{r+1} = k_p+1 \\ u_{r+1} \neq 0}} (1-q)^{\#(u_p, ..., u_r)} [\hat{K}, u_p, ..., u_{r+1}] & \text{if } \ s \leqslant p, \\ q^{r-s}[\hat{K}, k_p+1, 0, ..., 0] & \text{else,} \end{cases}$$

where $\#(u_p, ..., u_r)$ is the number of non-zero u_i.

Since $\boxdot_{\omega'}$ will never act on the last variable, we are only interested in the part of these sums such that $u_{r+1} = 1$. Let S be the part of $(1 + \mathbf{T}_r + \mathbf{T}_r\mathbf{T}_{r-1} + \cdots + \mathbf{T}_r \cdots \mathbf{T}_1)[\hat{K}, k_P+1]$ such that the exponent of the last variable is 1. The result of these computations is

$$S = (1 + q + \cdots + q^{p-1}) \sum_{u_p + \cdots + u_r = k_p} (1-q)^{\#(u_p, ..., u_r)} [\hat{K}, u_p, ..., u_r, 1].$$

By analogy, we recognize the sum

$$S = (1-q^p)(1 + \mathbf{T}_p + \mathbf{T}_{p+1}\mathbf{T}_p + \cdots + \mathbf{T}_{r-1} \cdots \mathbf{T}_p)[K, 0, ..., 0, 1].$$

Now using the facts that $\boxdot_i \mathbf{T}_i = q \boxdot_i$ and the factorization $\boxdot_{\omega'} = \boxdot_{\sigma'} \boxdot_i$, we find that $\boxdot_{\omega'} \mathbf{T}_i = q \boxdot_{\omega'}$, for any i. Hence

$$\boxdot_{\omega'} S = (1-q^p)(1 + q + \cdots + q^{r-p}) \boxdot_{\omega'}[K, 0, ..., 0, 1].$$

The coefficient $(1 + q + \cdots + q^{r-p})$ simplifies with the normalization factor of $G_{K \triangleright 1}$, so we have proved the lemma. ∎

As a consequence of the proof we get the expansions of the Hall–Littlewood basis in the quasi-monomial basis: Let $J \succcurlyeq I$ be two compositions. Let now $(k_1, ..., k_p)$ be the refining composition $\mathrm{Bre}(J, I)$. Then define

$$a_{I,J} = (1-q)^{k_1-1}(1-q^2)^{k_2-1}\cdots(1-q^p)^{k_p-1}. \tag{104}$$

The G_I expands

$$G_I = \sum_{J \succcurlyeq I} a_{I,J} M_J \tag{105}$$

The transition matrix is an upper unitriangular matrix (i.e., 1 on the diagonal, and $M_{I,J}$ is zero unless $I \succcurlyeq J$), corresponding to the inverse of the q-Kotska matrix. The analogue of the expression $s_\lambda = \sum K_{\lambda\mu}(q)\, P_\mu$ will be obtained by means of the dual basis, which lives in the space of noncommutative symmetric functions. We will prove that the coefficient of the expansion are polynomials in q with integer coefficients.

6.3. *Noncommutative Hall–Littlewood Functions*

In the former subsection we have proved that the family of quasi-symmetric Hall–Littlewood functions is a basis of **Qsym**. We can now define the noncommutative Hall–Littlewood symmetric functions by duality.

Definition 6.12. The elements of the space of noncommutative symmetric functions of the dual basis (H_K) of the (G_I) basis called noncommutative symmetric Hall–Littlewood functions.

The analogue of the expression $Q'_\mu = \sum K_{\lambda\mu} s_\lambda$ is the following formula.

Theorem 6.13. *The transition matrix whose rows are the H_J expanded in the ribbon basis (R_I) is a lower unitriangular matrix with positive coefficients. Moreover the expansion is given by*

$$H_K(A; q) = \sum_{K \succcurlyeq J} q^{t(K,J)} R_J, \tag{106}$$

where $t(K, J) = \mathrm{Maj}(\mathrm{Bre}(K, J)^\sim)$

Let us first explicit the coefficient $\mathrm{Maj}(\mathrm{Bre}(K, J)^\sim)$ for $K \succcurlyeq J$. Let us suppose that $K = (k_1, ..., k_p)$ and $J = (j_1, ..., j_q)$. Then by definition of the refinement order, there exist $0 < u_1 < u_2 < \cdots < u_q = p$ such that

$$J = (k_1 + k_2 + \cdots + k_{u_1}, k_{u_1+1} + \cdots + k_{u_1}, \cdots, k_{u_{p-1}+1} + \cdots + k_p).$$

Remark that the descent set of the composition $\mathrm{Bre}(K, J)$ is nothing but the set $U = \{u_1 < u_2 < \cdots < u_{q-1}\}$. Hence the descent set of the composition $\mathrm{Bre}(K, J)^{\sim}$ is the complementary of U in $\{1, ..., p-1\}$. Since the major index of a composition is the sum of its descents, the coefficient $t(K, J)$ is the sum of the positions of the "+" in the former expression of J. For example, if $K = (1, 3, 2, 1, 1, 3, 1, 4, 1)$ and $J = (4, 7, 1, 5)$, we write

$$rJ = (1+3,\ 2+1+1+3,\ 1,\ 4+1)$$

$$\text{positions:}\quad 1\ \ 2\ \ \ 3\ \ \ 4\ \ \ 5\ 6\ 7\ \ \ 8.$$

The + appear in positions 1, 3, 4, 5, and 8 so that $t(K, J) = 21$. We verify that $\mathrm{Bre}(K, J)^{\sim} = (2, 4, 1, 2)^{\sim} = (1, 2, 1, 1, 3, 1)$ and thus its major index is $5*1 + 4*2 + 3*1 + 2*1 + 3 = 21$ as well.

Proof. The ribbon basis (R_J) is the dual basis of the quasi-ribbon basis (F_J). Hence the duality pairing of G_I and H_K is

$$\begin{aligned}
\langle G_I, H_K\rangle &= \left\langle \sum_{J \succcurlyeq I} (-1)^{\ell(J)-\ell(I)}\, q^{s(I, J)} F_J\ \sum_{K \succcurlyeq L} q^{t(K, L)} R_L \right\rangle \\
&= \sum_{J \succcurlyeq I}\ \sum_{K \succcurlyeq L} (-1)^{\ell(J)-\ell(I)}\, q^{s(I, J)+t(K, L)} \langle F_J, R_L\rangle \\
&= \sum_{K \succcurlyeq J \succcurlyeq I} (-1)^{\ell(J)-\ell(I)} q^{s(I, J)+t(K, J)}.
\end{aligned} \tag{107}$$

Set $z(K, J, I) = (-1)^{\ell(J)-\ell(I)}\, q^{s(I, J)+t(K, J)}$. It is obvious that if $K = I$ the sum (107) reduces to 1. Hence, we only need to prove that the last sum is equal to zero for $K \succ I$.

Now suppose $I = (i_1, i_2, ..., i_p)$. Let l be the index of the first part of I which is broken in K, so that $K = (k_1, k_2, ..., k_q)$ with $i_u = k_u$ for $i < l$. Now the set $\{J \mid K \succcurlyeq J \succcurlyeq I\}$ breaks into two subsets, whether k_l appears alone as part of J or is added with k_{l+1}. Define the involution $\Psi_{K, I}$ of $\{J \mid K \succcurlyeq J \succcurlyeq I\}$ which exchanges the two preceding subsets as

$$\begin{aligned}
&\Psi_{K, I}(k_1, ..., k_{l-1}, j_l, ..., j_r) \\
&\qquad = \begin{cases} (k_1, ..., k_{l-1}, k_l + j_{l+1}, ..., j_r) & \text{if}\quad j_l = k_l \\ (k_1, ..., k_{l-1}, k_l, j_l - k_l, ..., j_r) & \text{if}\quad j_l > k_l. \end{cases}
\end{aligned}$$

For example, in the case $K = (2, 1, 1, 3, 2, 1, 1, 3)$ and $I = (2, 1, 6, 1, 2, 3)$, one has $l = 3$. Figure 3 shows the Hasse diagram of the set of composition J between K and I, together with the involution $\Psi_{K, I}$.

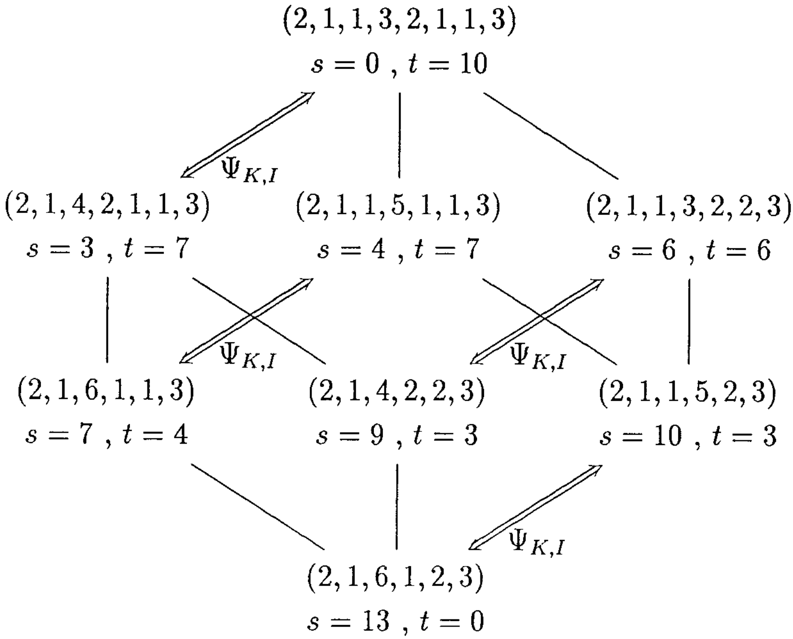


FIG. 3. The involution $\Psi_{K,I}$.

Now $\mathrm{Bre}(J, I)$ is of the form $(1^{l-1}, a, \ldots)$. Thus, if J is of the first kind $(j_l = k_l)$, the composition $\mathrm{Bre}(\Psi_{K,I}(J), I)$ is equal to $(1^{l-1}, a-1, \ldots)$. Therefore

$$s(I, \Psi_{K,I}(J)) = s(I, J) - q^l.$$

Similarly, we find that

$$t(I, \Psi_{K,I}(J)) = t(I, J) + q^l.$$

Hence

$$z(K, J, I) + z(K, \Psi_{K,I}(J), I) = 0$$

One verifies on Fig. 3 that the sum $s+t$ is constant along the double arrows corresponding to $\Psi_{K,I}$.

Since $\Psi_{K,I}$ is an involution, the sum (107) vanishes, and the proof is done. ∎

We observe that the analogues of the Kotska–Foulkes polynomials reduce here to power of q. For instance $H_{(1^n)} = \sum_K q^{\mathrm{Maj}(K^\sim)} R_K$ where $K^\sim$ is the conjugate composition of K. Another example is $H_{(3,2,1)} = R_{(3,2,1)} + qR_{(3,3)} + q^2 R_{(5,1)} + q^3 R_{(6)}$.

COROLLARY 6.14. *The H_I form a basis of noncommutative symmetric functions with coefficients in $\mathbb{Z}[q]$.*

The family $(G_K)_{(K)}$ is a basis of the space of quasi-symmetric functions with coefficient in $\mathbb{Z}[q]$.

Thus, we can express the product of two H_K.

THEOREM 6.15. *Let I and J two compositions of lengths r and s. Then*

$$H_I H_J = \sum_{J \succcurlyeq K} q^{t(J,K)} (c(r, s-p)\, H_{I \cdot K} + c(r, s-p+1)\, H_{I \triangleright K}), \tag{108}$$

where p is the length of the composition K and

$$c(r, v) = (1-q^r)(1-q^{r-1}) \cdots (1-q^{r-v+1}) \tag{109}$$

with the convention that $c(r, 0) = 1$ and $c(r, v) = 0$ if $v > i$.

This is a consequence of the formula $R_I R_J = R_{I \cdot J} + R_{I \triangleright J}$ (cf. [11]). For example,

$$\begin{aligned} H_{(3,1,2)} H_{(1,2)} = {} & H_{(3,1,2,1,2)} + (1-q^3)\, H_{(3,1,3,2)} \\ & + q(1-q^3)\, H_{(3,1,2,3)} + q(1-q^3)(1-q^2)\, H_{(3,1,5)}, \end{aligned}$$

and

$$\begin{aligned} H_{(1,1,1)} H_{(1,1)} = {} & H_{(1,1,1,1,1)} + (1-q^3)\, H_{(1,1,2,1)} \\ & + q(1-q^3)\, H_{(1,1,1,2)} + q(1-q^3)(1-q^2)\, H_{(1,1,3)}. \end{aligned}$$

Proof. On the two preceding examples, one can remark that the two coefficients are the same. This is a general fact, which gives the starting point of the proof.

The coefficients depends only on the lengths of the compositions I, J and on $\mathrm{Bre}(J, K)$, but not on the parts. Let us write this formally,

Suppose that $I' = (1^r)$ and $J' = (1^s)$. Let $U = (u_1, ..., u_{r+s}) = I \cdot J$ and $L = (l_1, ..., l_t)$ be a composition of $r+s$. To L we associate a composition $\phi_{(I,J)}(L)$ of $|I| + |J|$ obtained by summing the first l_1 parts of U, the next l_2 parts, and so one. That is, $\phi_{(I,J)}(L)$ is given by

$$(u_1 + u_2 + \cdots + u_{l_1 - 1}, u_{l_1} + \cdots + u_{l_1 + l_2 - 1}, ..., u_{l_1 + \cdots} + \cdots + u_{r+s}). \tag{110}$$

Then, $\phi_{(I,J)}(L)$ is a composition of the same length as L, and U is finer than $\phi_{(I,J)}(L)$. For example, with the previous notations, $I = (3, 1, 2)$,

$J=(1,2)$ so that $U=(3,1,2,1,2)$. Then $\phi_{(I,J)}(1,2,2)=(3,3,3)$ and $\phi_{(I,J)}(1,1,2,1)=(3,1,3,2)$.

The ribbon functions verify the equality

$$R_{\phi_{(I,J)}(U)} R_{\phi_{(I,J)}(V)} = R_{\phi_{(I,J)}(U \cdot V)} + R_{\phi_{(I,J)}(U \triangleright V)} \tag{111}$$

which says that the product of ribbon functions is "invariant under ϕ." Now, by definition of t, the coefficient of $H_{(1^r)} H_{(1^s)}$ on R_L is the same as the one of $H_I H_J$ on $R_{\phi_{(I,J)}(L)}$. Moreover, one has that $\mathrm{Bre}((1^{r+s}), L) = \mathrm{Bre}(U, \phi_{(I,J)}(L))$. It allows us to go back to the basis of Hall–Littlewood functions. Hence, we have proven the lemma:

Lemma 6.16. *The coefficient of $H_{(1^r)} H_{(1^s)}$ on H_L is the same as the one of $H_I H_J$ on $H_{\phi_{(I,J)}(L)}$.*

The coefficient of $H_I H_J$ on H_K where K is not of the form $\phi_{(I,J)}(L)$ is zero.

Hence it is enough to prove the case $H_{(1^r)} H_{(1^s)}$ for all r and s. Now suppose that K is a composition. Let us compute $H_{1^r \cdot K}$ and $H_{1^r \triangleright K}$. Recall that the coefficient $t(I, J)$ is obtained by adding the position of the $+$ when we write the parts of J as a sum of parts of I. There are two kinds of compositions L greater than $1^r \cdot K$:

- The ones for which r is a descent, that is, the compositions of the form $J \cdot K'$, where J is a composition of r and $K \succcurlyeq K'$. In this case the position of the $+$ smaller than r are the same in L and J and the positions greater than r in L are of the form $r+s$ where s is the position of the corresponding $+$ in K. Hence we get that

$$t(1^r \cdot K, J \cdot K') = t(1^r, J) + t(K, K') + (\ell(K) - \ell(K'))\, r. \tag{112}$$

For example, let $r=4$ and $K=(2,3,1,4,1)$. Then the composition $L=(2,1,1,5,1,5)$ is of the form (J, K'), with $J=(2,1,1) \models 4$ and $K \succcurlyeq K' = (5,1,5)$. Then one writes

$$J = (1+1,\ 1,\ 1,\ 2+3,\ 1,\ 4+1)$$

$$\text{positions:}\quad 1\ \ 2\ \ 3\ \ 4\quad 5\ \ 6\ \ 7\quad 8.$$

The positions of the $+$ are 1 corresponding to $J=(1+1,1,1)$ and 5, 8 which correspond to 1, 4 when writing $K'=(5,1,5)=(2+3,1,4+1)$. One can verify that $1+5+8=1+(1+5)+(5-3)\cdot 4$.

• The other compositions are the ones of the form $(J \rhd K')$, where J is a composition of r and $K \succcurlyeq K'$. The only difference with the preceding case is that there is a $+$ in the rth position. Hence

$$t(1^r \cdot K, J \rhd K') = t(1^r, J) + t(K, K') + (1 + \ell(K) - \ell(K'))\, r. \qquad (113)$$

For example, let $r=4$ and $K=(2,3,1,4,1)$. Then the composition $L=(2,1,6,1,5)$ is of the form $(J \rhd K')$, with $J=(2,1,1) \models 4$ and $K \succcurlyeq K' = (5,1,5)$. Then one writes

$$J = (1+1,\ 1,\ 1+2+3,\ 1,\ 4+1)$$

$$\text{positions:} \quad 1\ 2\ 3\quad 4\quad 5\ 6\ 7\quad 8.$$

One can verify that $1+4+5+8 = 1 + (1+5) + (5-3+1)*4$.

The result of these computations is that

$$H_{1^r \cdot K} = \sum_{J \models r} q^{t(1^r, J)} \sum_{K \succcurlyeq K'} q^{t(K, K') + (\ell(K) - \ell(K'))\, r} (R_{J \cdot K'} + q^r R_{J \rhd K'}). \qquad (114)$$

Similarly, one would find

$$H_{1^r \rhd K} = \sum_{J \models r} q^{t(1^r, J)} \sum_{K \succcurlyeq K'} q^{t(K, K') + (\ell(K) - \ell(K'))(r-1)} (R_{J \rhd K'}). \qquad (115)$$

But, from the expansion of H_I and the product rule of R_I we get

$$H_{(1^r)} H_{(1^s)} = \sum_{J \models r} q^{t(1^r, J)} \sum_{K' \models s} q^{t(1^s, K')} (R_{J \cdot K'} + R_{J \rhd K'}). \qquad (116)$$

We are trying to prove that

$$H_{(1^r)} H_{(1^s)} = \sum_{K \models s} q^{t(1^s, K)} (c(r, s - \ell(K))\, H_{1^r \cdot K} + (1 - q^r)\, c(r-1, s - \ell(K))\, H_{1^r \rhd K}). \qquad (117)$$

Hence we only need to prove the following formula corresponding to the coefficient of $R_{1^r \cdot K'}$,

$$q^{t(1^s, K')} = \sum_{(1^s) \succcurlyeq K \succcurlyeq K'} c(r, s - \ell(K))\, q^{(\ell(K) - \ell(K'))\, r} q^{t(1^s, K) + t(K, K')}. \qquad (118)$$

The coefficient of $R_{1^r \triangleright K'}$ follows immediately, replacing r by $r-1$. Let us prove this property by induction on K'. First define

$$z(1^s, K', K) = c(r, s-\ell(K))\, q^{(\ell(K)-\ell(K'))\, r} q^{t(1^s, K)+t(K, K')}. \tag{119}$$

If $K' = (1)$, and thus $s=1$, one has $z(1, 1, 1) = 1$ and the result is true. Moreover, if this is true for K' this is again true for $K' \cdot 1$, since $t(U \cdot 1, V \cdot 1)) = t(U, V)$ and thus

$$z(1^{s+1}, K' \cdot 1, K \cdot 1) = z(1^s, K', K). \tag{120}$$

Thus it remains to show that the property for K implies the property for the composition $K' \triangleright 1$. As usual the sum

$$\sum_{(1^{s+1}) \succcurlyeq K' \succcurlyeq K \triangleright 1} z(1^{s+1}, K', K \triangleright 1) \tag{121}$$

breaks into two parts depending on the fact that $K = L \cdot 1$ for some $L \models s$ or not ($K = L \triangleright 1$). Let us prove the following lemma

LEMMA 6.17. *Let* $(1^s) \succcurlyeq L \succcurlyeq K$. *Then one has*

$$\begin{aligned} z(1^{s+1}, L \cdot 1, K \triangleright 1) &= z(1^s, L, K)\, q^{r+\ell(L)} \\ z(1^{s+1}, L \triangleright 1, K \triangleright 1) &= z(1^s, L, K)\, q^s (1 - q^{r-s+\ell(L)}) \end{aligned} \tag{122}$$

In the first case $K = L \cdot 1$, one has $\ell(L) = \ell(K) + 1$, which gives the q^r. Equation (120) and the fact that

$$t(L \cdot 1, K' \triangleright 1) = t(L, K) + \ell(L) \tag{123}$$

give the required result. In the second case, one gets q^s from

$$t(1^{s+1}, L \triangleright 1) = t(1^s, L) + s \tag{124}$$

and the $(1 - q^{r-s+\ell(L)})$ comes from $c(r, s+1-\ell(L))$. This proves the lemma.

Finally one concludes that

$$\sum_{(1^{s+1}) \succcurlyeq K' \succcurlyeq K \triangleright 1} z(1^{s+1}, K', K \triangleright 1) = q^s \sum_{(1^s) \succcurlyeq K' \succcurlyeq K} z(1^s, K', K). \tag{125}$$

By induction the right hand side is equal to $q^s q^{t(1^s, K')}$, and the fact that

$$q^{t(1^{s+1}, K' \triangleright 1)} = q^s q^{t(1^s, K')} \tag{126}$$

gives the desired equality. The proof follows by induction. ∎

As a corollary we get the following striking property:

Corollary 6.18. *Let $H'_K = (1-q)^{\ell(K)} H_K$. Then the structure constants of* **Sym** *in the H'_K basis are polynomials in q with non-negative integer coefficients.*

Indeed, the preceding theorem shows that if I and J two compositions of lengths r and s, then

$$H'_I H'_J = \sum_{J \succcurlyeq K} q^{t(J, K)} (d(r, s-p) \, H'_{I \cdot K} + d(r, s-p+1) \, H'_{I \triangleright K}), \tag{127}$$

where p is the length of the composition K and

$$d(r, v) = [r]_q \, [r-1]_q \cdots [r-v+1]_q. \tag{128}$$

For example,

$$\begin{aligned} H'_{(3,1,2)} H'_{(1,2)} &= H'_{(3,1,2,1,2)} + (1+q+q^2) \, H'_{(3,1,3,2)} \\ &\quad + q(1+q+q^2) \, H'_{(3,1,2,3)} + q(1+q+q^2)(1+q) \, H'_{(3,1,5)}. \end{aligned}$$

This raises the question of an interpretation of these functions in the representation theory.

We are now interested in the specializations of the Hall–Littlewood functions at roots of unity. The noncommutative Hall–Littlewood functions have a factorization property similar the the one discovered by Lascoux, Leclerc, and Thibon [24, 25] (see also [38, 2]).

Corollary 6.19. *Let k be a integer and ζ be a kth root of the unity. Suppose that $I = (i_1, ..., i_p)$ is a composition. Writing $p = ck + r$, we break the composition I into blocks $I = J_1 \cdot J_2 \cdots J_{c+1}$ with $J_1, ..., J_c$ of length k and J_{c+1} of length r. Then the function $H_I(A; \zeta)$ factorizes in the following way:*

$$H_I(A; \zeta) = H_{J_1}(A; \zeta) \, H_{J_2}(A; \zeta) \cdots H_{J_{c+1}}(A; \zeta). \tag{129}$$

For instance, if ζ is a 3rd root of the unity,

$$H_{(3,2,4,1,5,3,2,1)}(A; \zeta) = H_{(3,2,4)}(A; \zeta) \, H_{(1,5,3)}(A; \zeta) \, H_{(2,1)}(A; \zeta).$$

Proof. This is an easy consequence of Theorem 6.15. By induction on c it is sufficient to prove that if J is a composition of length ck and K is a composition of any length one has $H_J(A;\zeta)\,H_K(A;\zeta)=H_{J,\,K}(A;\zeta)$, provided ζ is a kth root of the unity. But Theorem 6.15 reads

$$H_J(A;\zeta)\,H_K(A;\zeta)=H_{J\cdot K}(A;\zeta)+(1-\zeta^k)\left(\sum \text{other } H_L\right), \tag{130}$$

since $c(i,v)$ is a multiple of $(1-\zeta^k)$ unless $v=0$. So the proof is done. ∎

Note that this property remains true for the modified function H'_K.

APPENDIX A: TABLES

We give the transition matrices between the new Hall–Littlewood functions and the generalized Schur functions. The matrices are to be read as follows: The coefficient of the Hall–Littlewood function G_I (resp. H_I) on the function F_J (resp. R_J) is at the intersection of the row I and column J. A $\cdot$ means a zero coefficient. An integer i means q^i, with the convention that overlined numbers encode negative coefficients. For example, on the row indexed by 31 one reads that $G_{(31)}=F_{(31)}-qF_{(211)}-qF_{(121)}+q^2F_{(1111)}$.

$$\begin{array}{r|cccc|}
 & 3 & 21 & 12 & 111 \\
\hline
3 & 0 & \bar{1} & \bar{1} & 2 \\
21 & \cdot & 0 & \cdot & \bar{1} \\
12 & \cdot & \cdot & 0 & \bar{2} \\
111 & \cdot & \cdot & \cdot & 0
\end{array}$$

$M(G_I, F_J)$, degree 3

$$\begin{array}{r|cccc|}
 & 3 & 21 & 12 & 111 \\
\hline
3 & 0 & \cdot & \cdot & \cdot \\
21 & 1 & 0 & \cdot & \cdot \\
12 & 1 & \cdot & 0 & \cdot \\
111 & 3 & 1 & 2 & 0
\end{array}$$

$M(H_I, R_J)$, degree 3

$$\begin{array}{r|cccccccc|}
 & 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \\
\hline
4 & 0 & \bar{1} & \bar{1} & 2 & \bar{1} & 2 & 2 & \bar{3} \\
31 & \cdot & 0 & \cdot & \bar{1} & \cdot & \bar{1} & \cdot & 2 \\
22 & \cdot & \cdot & 0 & \bar{2} & \cdot & \cdot & \bar{1} & 3 \\
211 & \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \bar{1} \\
13 & \cdot & \cdot & \cdot & \cdot & 0 & \bar{2} & \bar{2} & 4 \\
121 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot & \bar{2} \\
112 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \bar{3} \\
1111 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0
\end{array}$$

$M(G_I, F_J)$, degree 4

$$\begin{array}{r|cccccccc|}
 & 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \\
\hline
4 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
31 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
22 & 1 & \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
211 & 3 & 1 & 2 & 0 & \cdot & \cdot & \cdot & \cdot \\
13 & 1 & \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\
121 & 3 & 1 & \cdot & \cdot & 2 & 0 & \cdot & \cdot \\
112 & 3 & \cdot & 1 & \cdot & 2 & \cdot & 0 & \cdot \\
1111 & 6 & 3 & 4 & 1 & 5 & 2 & 3 & 0
\end{array}$$

$M(H_I, R_J)$, degree 4

$$
\begin{array}{r}
\\
5\\ 41\\ 32\\ 311\\ 23\\ 221\\ 212\\ 2111\\ 14\\ 131\\ 122\\ 1211\\ 113\\ 1121\\ 1112\\ 11111
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccc}
5 & 41 & 32 & 311 & 23 & 221 & 212 & 2111 & 14 & 131 & 122 & 1211 & 113 & 1121 & 1112 & 11111
\end{array}\\
\left[\begin{array}{cccccccccccccccc}
0 & \bar{1} & \bar{1} & 2 & \bar{1} & 2 & 2 & \bar{3} & \bar{1} & 2 & 2 & \bar{3} & 2 & \bar{3} & \bar{3} & 4\\
. & 0 & . & \bar{1} & . & \bar{1} & . & 2 & . & \bar{1} & . & 2 & . & 2 & . & \bar{3}\\
. & . & 0 & \bar{2} & . & . & \bar{1} & 3 & . & . & \bar{1} & 3 & . & . & 2 & \bar{4}\\
. & . & . & 0 & . & . & . & \bar{1} & . & . & . & \bar{1} & . & . & . & 2\\
. & . & . & . & 0 & \bar{2} & \bar{2} & 4 & . & . & . & . & \bar{1} & 3 & 3 & \bar{5}\\
. & . & . & . & . & 0 & . & \bar{2} & . & . & . & . & . & \bar{1} & . & 3\\
. & . & . & . & . & . & 0 & \bar{3} & . & . & . & . & . & . & \bar{1} & 4\\
. & . & . & . & . & . & . & 0 & . & . & . & . & . & . & . & \bar{1}\\
. & . & . & . & . & . & . & . & 0 & \bar{2} & \bar{2} & 4 & \bar{2} & 4 & 4 & \bar{6}\\
. & . & . & . & . & . & . & . & . & 0 & . & \bar{2} & . & \bar{2} & . & 4\\
. & . & . & . & . & . & . & . & . & . & 0 & \bar{3} & . & . & \bar{2} & 5\\
. & . & . & . & . & . & . & . & . & . & . & 0 & . & . & . & \bar{2}\\
. & . & . & . & . & . & . & . & . & . & . & . & 0 & \bar{3} & \bar{3} & 6\\
. & . & . & . & . & . & . & . & . & . & . & . & . & 0 & . & \bar{3}\\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & \bar{4}\\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0
\end{array}\right]
\end{array}
$$

$M(G_I, F_J)$, degree 5

$$
\begin{array}{r}
\\
5\\ 41\\ 32\\ 311\\ 23\\ 221\\ 212\\ 2111\\ 14\\ 131\\ 122\\ 1211\\ 113\\ 1121\\ 1112\\ 11111
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccc}
5 & 41 & 32 & 311 & 23 & 221 & 212 & 2111 & 14 & 131 & 122 & 1211 & 113 & 1121 & 1112 & 11111
\end{array}\\
\left[\begin{array}{cccccccccccccccc}
0 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & .\\
1 & 0 & . & . & . & . & . & . & . & . & . & . & . & . & . & .\\
1 & . & 0 & . & . & . & . & . & . & . & . & . & . & . & . & .\\
3 & 1 & 2 & 0 & . & . & . & . & . & . & . & . & . & . & . & .\\
1 & . & . & . & 0 & . & . & . & . & . & . & . & . & . & . & .\\
3 & 1 & . & . & 2 & 0 & . & . & . & . & . & . & . & . & . & .\\
3 & . & 1 & . & 2 & . & 0 & . & . & . & . & . & . & . & . & .\\
6 & 3 & 4 & 1 & 5 & 2 & 3 & 0 & . & . & . & . & . & . & . & .\\
1 & . & . & . & . & . & . & . & 0 & . & . & . & . & . & . & .\\
3 & 1 & . & . & . & . & . & . & 2 & 0 & . & . & . & . & . & .\\
3 & . & 1 & . & . & . & . & . & 2 & . & 0 & . & . & . & . & .\\
6 & 3 & 4 & 1 & . & . & . & . & 5 & 2 & 3 & 0 & . & . & . & .\\
3 & . & . & . & 1 & . & . & . & 2 & . & . & . & 0 & . & . & .\\
6 & 3 & . & . & 4 & 1 & . & . & 5 & 2 & . & . & 3 & 0 & . & .\\
6 & . & 3 & . & 4 & . & 1 & . & 5 & . & 2 & . & 3 & . & 0 & .\\
10 & 6 & 7 & 3 & 8 & 4 & 5 & 1 & 9 & 5 & 6 & 2 & 7 & 3 & 4 & 0
\end{array}\right]
\end{array}
$$

$M(H_I, R_J)$, degree 5

$$
\begin{array}{r l}
 & \begin{array}{cccccccccccccccccccccccccccccccc} 6 & 51 & 42 & 411 & 33 & 321 & 312 & 3111 & 24 & 231 & 222 & 2211 & 213 & 2121 & 2112 & 21111 & 15 & 141 & 132 & 1311 & 123 & 1221 & 1212 & 12111 & 114 & 1131 & 1122 & 11211 & 1113 & 11121 & 11112 & 111111 \end{array} \\
\begin{array}{r} 6 \\ 51 \\ 42 \\ 411 \\ 33 \\ 321 \\ 312 \\ 3111 \\ 24 \\ 231 \\ 222 \\ 2211 \\ 213 \\ 2121 \\ 2112 \\ 21111 \\ 15 \\ 141 \\ 132 \\ 1311 \\ 123 \\ 1221 \\ 1212 \\ 12111 \\ 114 \\ 1131 \\ 1122 \\ 11211 \\ 1113 \\ 11121 \\ 11112 \\ 111111 \end{array} &
\left[\begin{array}{cccccccccccccccccccccccccccccccc}
0 & \bar{1} & \bar{1} & 2 & \bar{1} & 2 & 2 & \bar{3} & \bar{1} & 2 & 2 & \bar{3} & 2 & \bar{3} & \bar{3} & 4 & \bar{1} & 2 & 2 & \bar{3} & 2 & \bar{3} & \bar{3} & 4 & 2 & \bar{3} & \bar{3} & 4 & \bar{3} & 4 & 4 & \bar{5} \\
. & 0 & . & \bar{1} & . & \bar{1} & . & 2 & . & \bar{1} & . & 2 & . & 2 & . & \bar{3} & . & \bar{1} & . & 2 & . & 2 & . & \bar{3} & . & 2 & . & \bar{3} & . & \bar{3} & . & 4 \\
. & . & 0 & \bar{2} & . & . & \bar{1} & 3 & . & . & \bar{1} & 3 & . & . & 2 & \bar{4} & . & . & \bar{1} & 3 & . & . & 2 & \bar{4} & . & . & 2 & \bar{4} & . & . & \bar{3} & 5 \\
. & . & . & 0 & . & . & . & \bar{1} & . & . & . & \bar{1} & . & . & . & 2 & . & . & . & \bar{1} & . & . & . & 2 & . & . & . & 2 & . & . & . & \bar{3} \\
. & . & . & . & 0 & \bar{2} & \bar{2} & 4 & . & . & . & . & \bar{1} & 3 & 3 & \bar{5} & . & . & . & . & \bar{1} & 3 & 3 & \bar{5} & . & . & . & . & 2 & \bar{4} & \bar{4} & 6 \\
. & . & . & . & . & 0 & . & \bar{2} & . & . & . & . & . & \bar{1} & . & 3 & . & . & . & . & . & \bar{1} & . & 3 & . & . & . & . & . & 2 & . & \bar{4} \\
. & . & . & . & . & . & 0 & \bar{3} & . & . & . & . & . & . & \bar{1} & 4 & . & . & . & . & . & . & \bar{1} & 4 & . & . & . & . & . & . & 2 & \bar{5} \\
. & . & . & . & . & . & . & 0 & . & . & . & . & . & . & . & \bar{1} & . & . & . & . & . & . & . & \bar{1} & . & . & . & . & . & . & . & 2 \\
. & . & . & . & . & . & . & . & 0 & \bar{2} & \bar{2} & 4 & \bar{2} & 4 & 4 & \bar{6} & . & . & . & . & . & . & . & . & \bar{1} & 3 & 3 & \bar{5} & 3 & \bar{5} & \bar{5} & 7 \\
. & . & . & . & . & . & . & . & . & 0 & . & \bar{2} & . & \bar{2} & . & 4 & . & . & . & . & . & . & . & . & . & \bar{1} & . & 3 & . & 3 & . & \bar{5} \\
. & . & . & . & . & . & . & . & . & . & 0 & \bar{3} & . & . & \bar{2} & 5 & . & . & . & . & . & . & . & . & . & . & \bar{1} & 4 & . & . & 3 & \bar{6} \\
. & . & . & . & . & . & . & . & . & . & . & 0 & . & . & . & \bar{2} & . & . & . & . & . & . & . & . & . & . & . & \bar{1} & . & . & . & 3 \\
. & . & . & . & . & . & . & . & . & . & . & . & 0 & \bar{3} & \bar{3} & 6 & . & . & . & . & . & . & . & . & . & . & . & . & \bar{1} & 4 & 4 & \bar{7} \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 0 & . & \bar{3} & . & . & . & . & . & . & . & . & . & . & . & . & . & \bar{1} & . & 4 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & \bar{4} & . & . & . & . & . & . & . & . & . & . & . & . & . & . & \bar{1} & 5 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & \bar{1} \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & \bar{2} & \bar{2} & 4 & \bar{2} & 4 & 4 & \bar{6} & \bar{2} & 4 & 4 & \bar{6} & 4 & \bar{6} & \bar{6} & 8 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & . & \bar{2} & . & \bar{2} & . & 4 & . & \bar{2} & . & 4 & . & 4 & . & \bar{6} \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & \bar{3} & . & . & \bar{2} & 5 & . & . & \bar{2} & 5 & . & . & 4 & \bar{7} \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & . & . & . & \bar{2} & . & . & . & \bar{2} & . & . & . & 4 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & \bar{3} & \bar{3} & 6 & . & . & . & . & \bar{2} & 5 & 5 & \bar{8} \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & . & \bar{3} & . & . & . & . & . & \bar{2} & . & 5 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & \bar{4} & . & . & . & . & . & . & \bar{2} & 6 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & . & . & . & . & . & . & . & \bar{2} \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & \bar{3} & \bar{3} & 6 & \bar{3} & 6 & 6 & \bar{9} \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & . & \bar{3} & . & \bar{3} & . & 6 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & \bar{4} & . & . & \bar{3} & 7 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & . & . & . & \bar{3} \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & \bar{4} & \bar{4} & 8 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & . & \bar{4} \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & \bar{5} \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0
\end{array}\right]
\end{array}
$$

$M(G_I, F_J)$, degree 6

	6	51	42	411	33	321	312	3111	24	231	222	2211	213	2121	2112	21111	15	141	132	1311	123	1221	1212	12111	114	1131	1122	11211	1113	11121	11112	111111
6	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
51	1	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
42	1	.	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
411	3	1	2	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
33	1	.	.	.	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
321	3	1	.	.	2	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
312	3	.	1	.	2	.	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3111	6	3	4	1	5	2	3	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
24	1	.	.	.	.	.	.	.	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
231	3	1	.	.	.	.	.	.	2	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
222	3	.	1	.	.	.	.	.	2	.	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2211	6	3	4	1	.	.	.	.	5	2	3	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
213	3	.	.	.	1	.	.	.	2	.	.	.	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2121	6	3	.	.	4	1	.	.	5	2	.	.	3	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2112	6	.	3	.	4	.	1	.	5	.	2	.	3	.	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
21111	10	6	7	3	8	4	5	1	9	5	6	2	7	3	4	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
15	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
141	3	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.
132	3	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.	0	.	.	.	.	.	.	.	.	.	.	.	.	.
1311	6	3	4	1	.	.	.	.	.	.	.	.	.	.	.	.	5	2	3	0	.	.	.	.	.	.	.	.	.	.	.	.
123	3	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	2	.	.	.	0	.	.	.	.	.	.	.	.	.	.	.
1221	6	3	.	.	4	1	.	.	.	.	.	.	.	.	.	.	5	2	.	.	3	0	.	.	.	.	.	.	.	.	.	.
1212	6	.	3	.	4	.	1	.	.	.	.	.	.	.	.	.	5	.	2	.	3	.	0	.	.	.	.	.	.	.	.	.
12111	10	6	7	3	8	4	5	1	.	.	.	.	.	.	.	.	9	5	6	2	7	3	4	0	.	.	.	.	.	.	.	.
114	3	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	2	.	.	.	.	.	.	.	0	.	.	.	.	.	.	.
1131	6	3	.	.	.	.	.	.	4	1	.	.	.	.	.	.	5	2	.	.	.	.	.	.	3	0	.	.	.	.	.	.
1122	6	.	3	.	.	.	.	.	4	.	1	.	.	.	.	.	5	.	2	.	.	.	.	.	3	.	0	.	.	.	.	.
11211	10	6	7	3	.	.	.	.	8	4	5	1	.	.	.	.	9	5	6	2	.	.	.	.	7	3	4	0	.	.	.	.
1113	6	.	.	.	3	.	.	.	4	.	.	.	1	.	.	.	5	.	.	.	2	.	.	.	3	.	.	.	0	.	.	.
11121	10	6	.	.	7	3	.	.	8	4	.	.	5	1	.	.	9	5	.	.	6	2	.	.	7	3	.	.	4	0	.	.
11112	10	.	6	.	7	.	3	.	8	.	4	.	5	.	1	.	9	.	5	.	6	.	2	.	7	.	3	.	4	.	0	.
111111	15	10	11	6	12	7	8	3	13	8	9	4	10	5	6	1	14	9	10	5	11	6	7	2	12	7	8	3	9	4	5	0

$M(H_I, R_J)$, degree 6

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